

# On the complexity of Temporal Equilibrium Logic

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**Abstract**—Temporal Equilibrium Logic (TEL) [1] is a promising framework that extends the knowledge representation and reasoning capabilities of Answer Set Programming with temporal operators in the style of LTL. To our knowledge it is the first nonmonotonic logic that accommodates fully the syntax of a standard temporal logic (specifically LTL) without requiring further constructions. This paper provides a systematic complexity analysis for the (consistency) problem of checking the existence of a temporal equilibrium model of a TEL formula. It was previously shown that this problem in the general case lies somewhere between PSPACE and EXPSpace. Here we establish a lower bound matching the EXPSpace upper bound in [2]. Additionally we analyse the complexity for various natural subclasses of TEL formulas, identifying both tractable and intractable fragments. Finally the paper offers some new insights on the logic LTL by addressing satisfiability for minimal LTL models. The complexity results obtained highlight a substantial difference between interpreting LTL over finite or infinite words.

## I. INTRODUCTION

In this paper we analyse the complexity of checking model existence in Temporal Equilibrium Logic (TEL). TEL was proposed by Cabalar and Vega [1] as a nonmonotonic logic for temporal reasoning. In particular, TEL provides an important extension of the language of answer set programming (ASP) by capturing temporal reasoning problems not representable in ASP. It is also apparently the only nonmonotonic extension of a standard modal temporal logic (viz. LTL) that does not use additional operators or constructions.

Answer Set Programming (ASP) is now well established as a successful paradigm for declarative programming, with its roots in the fields of knowledge representation (KR), logic programming, and nonmonotonic reasoning (NMR) [3]. Besides a fully declarative, modular reading of problem descriptions, distinguishing features of ASP are its intrinsic handling of nondeterminism and the rich possibilities for knowledge representation, including the seamless handling of incomplete and defeasible knowledge, preferences at various levels, as well as aggregates and other useful features.

An adequate logical foundation for ASP is provided by a formalism called *Equilibrium Logic* [4], [5], a nonmonotonic extension of the superintuitionistic logic of *here-and-there* (HT) [6]. This provides useful logical tools for the metatheory of ASP and a framework for defining extensions of the basic ASP language, for example to arbitrary propositional and first-order theories, to languages with intensional functions, and to hybrid theories that combine classical and rule-based reasoning [7]–[10].

The nonmonotonic capability of ASP helps to solve typical representation issues in temporal reasoning such as the frame

problem [11] and the ramification problem [12]. However, while ASP has been applied to a wide range of problems involving temporal reasoning, including prediction, planning, diagnosis and verification, since it is not an intrinsically temporal formalism, it suffers some important limitations. Most ASP solvers deal with finite domains, a restriction that allows a grounding of the program into a finite set of propositional rules. This limitation means that time is usually represented by an extensional predicate with a finite domain fixed *a priori*, hampering the solution of problems dealing with unbounded time.

Temporal scenarios dealing with unbounded time are typically best suited for modal temporal logics. However, standard modal temporal logics, such as propositional linear-time temporal logic LTL [13], do not accommodate default and nonmonotonic reasoning and are not designed to deal with many issues in knowledge representation. TEL extends equilibrium logic and therefore includes KR features from ASP but is able to express concepts from modal temporal logic. It shares the syntax of LTL, but its semantics is an orthogonal combination of the LTL semantics with the nonmonotonic semantics of Equilibrium Logic. As for Equilibrium Logic, TEL models (called *temporal equilibrium models*) are the result of a kind of minimisation among models of the monotonic logic of Temporal Here-and-There (THT), a combination of LTL and HT. Considerable progress has already been made in the theoretical study of TEL and its computational methods. Key results include the use of TEL to translate action languages [1], an automata-theoretic approach for checking the existence of TEL models [2], a decidable criterion for proving the strong equivalence of two TEL theories [14], and a tool for computing models of temporal programs under TEL semantics [15].

**Our contribution:** We investigate the computational cost of the TEL consistency problem, that is checking for a given THT formula the existence of a temporal equilibrium model. This question was previously addressed in [2] by showing that the problem lies somewhere between PSPACE and EXPSpace. Our first contribution consists in filling this computational gap by providing a lower bound matching the EXPSpace upper bound in [2].

As a second contribution, we give a systematic analysis, searching for natural subclasses of THT formulas for which complexity decreases. In particular, we consider all the syntactical fragments of THT obtained by restricting the set of allowed temporal modalities and/or by imposing a bound on the nesting depth of temporal modalities and/or the implication connective (including negation, expressed in terms of implica-

tion). The aim is to obtain a better understanding of what makes the initial problem EXPSPACE-hard, and to identify interesting fragments with lower complexity. Overall, our results are rather negative. We show that the TEL consistency problem remains EXPSPACE-hard even in the following two simple cases: (1) the unique allowed temporal modality is  $G$  ('always'), and (2) there is no nesting of implication.

The result for the first case is surprising since LTL/THT satisfiability for the fragment where the unique allowed temporal modalities are  $G$  and  $F$  ('eventually') is just NP-complete [2], [16]. On the other hand, the result for the second case highlights an important difference between propositional equilibrium logic and TEL. It is well-known that for logic programs without default negation (corresponding to HT formulas where there is no nesting of implication<sup>1</sup>), the existence of classical models ensures the existence of stable models. This fails in the temporal extension, where as pointed in [2], the non-existence of equilibrium models may be also due to the lack of a finite justification for satisfying the criterion of minimal knowledge.

The TEL consistency problem remains hard, and, precisely, NEXPTIME-complete even for the simple case where no nesting of temporal modalities is allowed. However, on the positive side, we identify many interesting THT fragments with a lower complexity. For each of them, we show that the TEL consistency problem is complete for some complexity class in  $\{NP, \Sigma_2, PSPACE\}$  (for an overview of the obtained results, see Subsection II-A). Some of these results also point out a peculiar difference between LTL and THT: due to the interpretation of the implication connective, in THT, a temporal modality cannot be expressed in terms of its 'dual' modality. Thus, in THT, dual temporal modalities, such as  $F$  and  $G$ , need to be considered independently from one another. This is illustrated by one of our positive results: for the THT fragment whose allowed temporal modalities are  $F$  and  $X$  ('next'), the complexity of the considered problem collapses to the second level  $\Sigma_2$  of the polynomial hierarchy. This also turns out to be the unique case where, surprisingly, LTL/THT satisfiability is harder than TEL consistency.

As a third contribution, we provide new insights into the logic LTL. We address minimal LTL satisfiability, that is checking the existence of LTL models which are minimal with respect to the partial order given by pointwise propositional containment. While for LTL over finite words, the existence of LTL models ensures the existence of minimal ones, for LTL over infinite words, this is not true. In particular, we show that for the case of infinite words, minimal LTL satisfiability is exponentially harder than LTL satisfiability, and, precisely, EXPSPACE-complete. To the best of our knowledge, there is no complexity result in the literature emphasizing the differences arising from interpreting LTL over finite or infinite words.

*Related work:* Several research areas of AI have combined modal temporal logics with formalisms from knowledge representation for reasoning about actions and planning (see e.g. [17]). Combinations of NMR with modal logics

designed for temporal reasoning are much more infrequent in the literature. The few exceptions are typically modal action languages with a nonmonotonic semantics defined under some syntactical restrictions. Recently, an alternative to TEL has been introduced, namely, Temporal Answer Sets (TAS), which relies on dynamic linear-time temporal logic [18], a modal approach more expressive than LTL. However, while the non-monotonic semantics of TEL covers any arbitrary theory in the syntax of LTL, TAS uses a syntactic transformation that is only defined for theories with a rather restricted syntax. A framework unifying TEL and TAS has been proposed in [19].

## II. TEMPORAL EQUILIBRIUM LOGIC

We recall the framework of Temporal Equilibrium Logic (TEL) [1]. TEL is defined by first introducing a monotonic and intermediate version of standard linear temporal logic LTL [13], the so-called logic of *Temporal Here-and-There* (THT) [1]. The nonmonotonic semantics of TEL is then defined by introducing a criterion for selecting models of THT.

Let  $\mathbb{N}$  be the set of natural numbers and for all  $i, j \in \mathbb{N}$ , let  $[i, j] := \{h \in \mathbb{N} \mid i \leq h \leq j\}$ . For an infinite word  $w$  over some alphabet and for all  $i \geq 0$ ,  $w(i)$  is the  $i^{\text{th}}$  symbol of  $w$ .

*Syntax and semantics of THT:* while the syntax of THT coincides with that of LTL, the semantics of THT is instead an orthogonal combination of the superintuitionistic propositional logic of Here-and-There (HT) [6] and LTL. Fix a finite set  $P$  of atomic propositions. The set of THT formulas  $\varphi$  over  $P$  is defined by the following abstract syntax.

$$\varphi := p \mid \perp \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \varphi \rightarrow \varphi \mid X\varphi \mid \varphi U \varphi \mid \varphi R \varphi$$

where  $p \in P$  and  $X$ ,  $U$ , and  $R$ , are the standard 'next', 'until', and 'release' temporal modalities. Negation is defined as  $\neg\varphi \stackrel{\text{def}}{=} \varphi \rightarrow \perp$  while  $\top \stackrel{\text{def}}{=} \neg\perp$ . As usual  $\varphi_1 \leftrightarrow \varphi_2$  stands for  $(\varphi_1 \rightarrow \varphi_2) \wedge (\varphi_2 \rightarrow \varphi_1)$ . The classical temporal operators  $G$  ('always') and  $F$  ('eventually') can be defined in terms of  $U$  and  $R$  as follows:  $F\varphi \stackrel{\text{def}}{=} \top U \varphi$  and  $G\varphi \stackrel{\text{def}}{=} \perp R \varphi$ . The size  $|\varphi|$  of a formula  $\varphi$  is the number of distinct subformulas of  $\varphi$ . The *temporal height* (resp. *implication height*) of  $\varphi$  is the maximum number of nested temporal modalities (resp. nested implications) in  $\varphi$ . Notice that negation is counted as an additional implication. Thus, for example, formula  $\neg p \rightarrow p$  has implication height equal to 2.

Recall that LTL over  $P$  is interpreted on infinite words over  $2^P$ , called in the following *LTL interpretations*. By contrast, the semantics of THT is defined in terms of infinite words over  $2^P \times 2^P$ , which can also be viewed as pairs of LTL-interpretations. Formally, a *THT interpretation* is a pair  $M = (H, T)$  consisting of two LTL interpretations:  $H$  (the 'here' interpretation) and  $T$  (the 'there' interpretation) such that

$$\text{for all } i \geq 0, H(i) \subseteq T(i)$$

Intuitively,  $H(i)$  represents the set of propositions which are true at position  $i$ , while  $T(i)$  is the set of propositions which *may* be true (i.e. which are not falsified in an intuitionistic sense). A THT interpretation  $M = (H, T)$  is said to be *total*

<sup>1</sup>recall that in HT/THT negation is expressed in terms of implication

whenever  $H = T$ . In the following, for *interpretation*, we mean a THT interpretation. Given an interpretation  $M = (H, T)$ , a position  $i \geq 0$ , and a THT formula  $\varphi$ , the satisfaction relation  $M, i \models \varphi$  is inductively defined as follows:

$$\begin{aligned}
M, i &\not\models \perp \\
M, i &\models p &\Leftrightarrow p \in H(i) \\
M, i &\models \varphi \vee \psi &\Leftrightarrow \text{either } M, i \models \varphi \text{ or } M, i \models \psi \\
M, i &\models \varphi \wedge \psi &\Leftrightarrow M, i \models \varphi \text{ and } M, i \models \psi \\
M, i &\models \varphi \rightarrow \psi &\Leftrightarrow \text{for all } H' \in \{H, T\}, \\
&&\quad \text{either } (H', T), i \not\models \varphi \text{ or } (H', T), i \models \psi \\
M, i &\models X\varphi &\Leftrightarrow M, i+1 \models \varphi \\
M, i &\models \varphi U \psi &\Leftrightarrow \text{there is } j \geq i \text{ such that } M, j \models \psi \text{ and} \\
&&\quad \text{for all } k \in [i, j-1], M, k \models \varphi \\
M, i &\models \varphi R \psi &\Leftrightarrow \text{for all } j \geq i, \text{ either } M, j \models \psi \text{ or there} \\
&&\quad \text{is } k \in [i, j-1] \text{ such that } M, k \models \varphi
\end{aligned}$$

We say that  $M$  is a (THT) model of  $\varphi$ , written  $M \models \varphi$ , whenever  $M, 0 \models \varphi$ . A THT formula  $\varphi$  is THT *satisfiable* if it admits a THT model. A formula  $\varphi$  is THT *valid* if every interpretation  $M$  is a THT model of  $\varphi$ . Note that the semantics of THT is defined similarly to that of LTL except for the clause for the implication connective  $\rightarrow$  which must be checked in both the components  $H$  and  $T$  of  $M$ . As a consequence  $M, i \not\models \varphi$  does not correspond to  $M, i \models \neg\varphi$  (i.e.,  $M, i \models \neg\varphi$  implies that  $M, i \not\models \varphi$ , but the converse direction does not hold in general). However, if we restrict the semantics to total interpretations,  $(T, T) \models \varphi$  corresponds to the satisfaction relation  $T \models \varphi$  in LTL. More precisely, the LTL models  $T$  of  $\varphi$  correspond to the total interpretations  $(T, T)$  which are THT models of  $\varphi$ . As shown in [2], THT satisfiability can be reduced in linear-time to LTL satisfiability. With regard to THT validity, a THT valid formula is also an LTL valid formula, but the converse in general does not hold. For example, the *excluded middle* axiom  $\varphi \vee \neg\varphi$  is not a valid THT formula since, as highlighted above, for an interpretation  $M = (H, T)$ ,  $M \not\models \varphi$  does not imply that  $M \models \neg\varphi$ . Similarly, the temporal formulas  $F\varphi \leftrightarrow \neg G\neg\varphi$  and  $\varphi_1 U \varphi_2 \leftrightarrow \neg\varphi_1 R \neg\varphi_2$ , which are well-known valid LTL formulas (and allow to express, in LTL, a temporal modality in terms of its dual modality), are not THT valid formulas. Thus, in THT, dual temporal modalities, like  $F$  and  $G$ , or  $U$  and  $R$ , need to be considered independently one from the other one. We summarize some observations made above and some additional observations (which easily follows from the semantics of THT and LTL) in the following proposition, where for clarity, we use  $\models_{\text{LTL}}$  to denote the satisfaction relation in LTL.

**Proposition II.1.** *Let  $(H, T)$  be an interpretation and  $\varphi$  be a THT formula.*

- 1) *If  $(H, T), i \models \varphi$ , then  $(T, T), i \models \varphi$  (for all  $i \geq 0$ ).*
- 2)  *$(H, T), i \models \neg\varphi$  iff  $(T, T), i \models \neg\varphi$  (for all  $i \geq 0$ ).*
- 3)  *$(T, T) \models \varphi$  iff  $T \models_{\text{LTL}} \varphi$ .*
- 4) *If  $\varphi$  has implication height at most 1, then  $(H, T) \models \varphi$  implies  $H \models_{\text{LTL}} \varphi$ .*

*The non-monotonic logic TEL:* this logic is obtained from THT by restricting the semantics to a subclass of models,

called *temporal equilibrium models*. For two LTL interpretations  $H$  and  $T$ , we write  $H \sqsubseteq T$  to mean that  $H(i) \subseteq T(i)$  for all  $i \geq 0$ . We write  $H \sqsubset T$  to mean that  $H \sqsubseteq T$  and  $H \neq T$ .

**Definition II.1** (Temporal equilibrium model). *Given a THT formula  $\varphi$ , a (temporal) equilibrium model of  $\varphi$  is a total model  $(T, T)$  of  $\varphi$  satisfying the following minimality requirement: whenever  $H \sqsubset T$ , then  $(H, T) \not\models \varphi$ .*

If we restrict the syntax to HT formulas (i.e., THT formulas where no temporal modality is allowed) and the semantics to HT interpretations  $(H(0), T(0))$ , then (non-temporal) equilibrium models coincide with stable models of answer set programs in their most general form [20]. In particular, the interpretation of negation is that of default negation in logic programming: formula  $\neg\varphi$  holds ( $\varphi$  is false by default) if there is no evidence regarding  $\varphi$ , i.e.,  $\varphi$  cannot be derived by the rules of the logic program. As an example, let us consider the THT formula  $\varphi$  given by  $\varphi = G(\neg p \rightarrow Xp)$ . Its intuitive meaning corresponds to the first-order logic program consisting of rules of the form  $p(s(X)) \leftarrow \text{not } p(X)$ , where time has been reified as an extra parameter  $X = 0, s(0), s(s(0)), \dots$ . Thus, at any time instant, if there is no evidence regarding  $p$ , then  $p$  will become true at the next instant. Initially, we have no evidence regarding  $p$ , so this will imply  $Xp$ . To derive  $XXp$ , the only possibility would be the rule  $\neg Xp \rightarrow XXp$ , an instance of  $\varphi$ . As the body of this rule is false,  $XXp$  becomes false by default, and so on. It is easy to see that the unique equilibrium model of  $\varphi$  is  $((\emptyset\{p\})^\omega, (\emptyset\{p\})^\omega)$ , corresponding to the unique LTL model of formula  $\neg p \wedge G(\neg p \leftrightarrow Xp)$ .

Note that an LTL satisfiable formula may have no temporal stable model. A familiar example from non-temporal ASP is the logic program rule  $\neg p \rightarrow p$ , whose unique classical model is  $\{p\}$  and whose HT models are  $(\emptyset, \{p\})$  and  $(\{p\}, \{p\})$ . As a second example, consider the temporal formula  $\varphi$  given by  $\varphi = G(\neg Xp \rightarrow p) \wedge G(Xp \rightarrow p)$ . This formula is LTL-equivalent to  $Gp$ . Thus, the unique LTL model is  $T = \{p\}^\omega$ . However,  $(T, T)$  is not an equilibrium model of  $\varphi$ , since the interpretation  $(H, T)$ , where  $H = (\emptyset)^\omega$  is a THT model of  $\varphi$ .

In general, for HT formulas, the non-existence of equilibrium models is due to the unrestricted use of nested implication (recall that negation is expressed in terms of implication). For the temporal case, as pointed in [2], the non-existence of equilibrium models may be also due to the lack of a finite justification which ensures the minimal fulfilment of the given formula. For example, for the formula  $\varphi = GFp$ , any LTL model  $T$  must contain infinite occurrences of  $p$  (hence, no prefix of  $T$  can justify the fulfilment of  $\varphi$ ). Even if  $\varphi$  is THT/LTL satisfiable, one can easily check that there is no equilibrium model of  $\varphi$ .

#### A. Summary of the results

We are interested in the following decision problem.

*The TEL consistency problem::* let  $\mathcal{L}$  be THT or a fragment of THT. The TEL consistency decision problem for  $\mathcal{L}$ , written  $\text{CON}(\mathcal{L})$ , is the set of all  $\mathcal{L}$ -formulas for which there exists an equilibrium model.

TABLE I  
COMPUTATIONAL COST OF THE TEL CONSISTENCY PROBLEM

$m \geq 1, k \geq 1$	TEL consistency problem
THT, $\text{THT}_{m+1}^1(\text{F}, \text{G})$ , $\text{THT}_{m+1}^{k+1}(\text{G})$ , $\text{THT}_{m+1}^{k+1}(\text{U})$	EXPSpace-complete (Theorem III.1 and [2])
THT(X), THT(F), THT(X,F)	$\Sigma_2$ -complete (Corollary IV.2)
$\text{THT}_1$ , $\text{THT}_1^{k+1}(\text{F}, \text{G})$ , $\text{THT}_1^{k+1}(\text{U})$ , $\text{THT}_1^{k+1}(\text{R})$	NEXPTIME-complete (Theorems III.2 and III.3)
$\text{THT}_1(\text{X}, \text{G})$	$\Sigma_2$ -complete (Theorem IV.2)
$\text{THT}^1(\text{R})$ , $\text{THT}^1(\text{X}, \text{R})$ , $\text{THT}^1(\text{U})$ , $\text{THT}^1(\text{X}, \text{U})$	PSPACE-complete (Theorem IV.4 and Cor. IV.1)
$\text{THT}_1^1$	NP-complete (Theorem IV.4)
$\text{THT}^0$	PSPACE-hard (Theorem IV.1)
$\text{THT}_0 = \text{HT}$	$\Sigma_2$ -complete [5], [21]

In particular, we consider the syntactical fragments of THT obtained by restricting the set of allowed temporal modalities and/or by bounding the temporal/implication height. Formally, given  $O_1, O_2, \dots \in \{\text{X}, \text{F}, \text{G}, \text{U}, \text{R}\}$ , we denote by  $\text{THT}(O_1, O_2, \dots)$  the fragment of THT for which only the temporal modalities  $O_1, O_2, \dots$  are allowed. For  $k \geq 0$  and  $m \geq 0$ ,  $\text{THT}_m^k(O_1, O_2, \dots)$  denotes the fragment of  $\text{THT}(O_1, O_2, \dots)$  where the temporal height is at most  $m$  and the implication height is at most  $k$ . We write nothing for  $m$  and/or  $k$  when no bound is imposed. For instance,  $\text{THT}_2(\text{G})$  denotes the fragment where the unique allowed temporal modality is  $\text{G}$  and the temporal height is at most 2. The results obtained in this paper are illustrated in Fig. I. Notice that  $\text{THT}_0 = \text{HT}$  and checking the existence of equilibrium models for HT formulas is a well-known  $\Sigma_2$ -complete problem [5], [21]. Moreover, membership in EXPSpace for the TEL consistency problem of full THT has been established in [2] by a generalisation of the standard automata-theoretic approach for solving LTL satisfiability.

Additionally, in Section V, we investigate the complexity of checking for a given THT formula  $\varphi$ , the existence of a *minimal LTL model*, i.e. an LTL model  $\text{T}$  of  $\varphi$  such that for all  $\text{H} \sqsubset \text{T}$ ,  $\text{H} \not\models_{\text{LTL}} \varphi$ . Notice that in general LTL satisfiability does not ensure the existence of minimal LTL models. An example is given by the formula  $\text{GF}p$  which is LTL satisfiable but does not admit minimal LTL models.

### III. INTRACTABLE FRAGMENTS

In this section we show that the TEL consistency problem is in general EXPSpace-hard even for the fragments  $\text{THT}_2^1(\text{F}, \text{G})$ ,  $\text{THT}_2^2(\text{G})$ , and  $\text{THT}_2^2(\text{U})$ . Moreover, the problem remains hard, and, precisely, NEXPTIME-complete when no nesting of temporal modalities is allowed. Notice that EXPSpace-hardness for  $\text{THT}_2^1(\text{F}, \text{G})$  is surprising since THT satisfiability for the fragment  $\text{THT}(\text{F}, \text{G})$  is just NP-complete [2], [16] and checking the existence of equilibrium models for  $\text{HT}^1$  formulas has the same complexity as satisfiability of classical propositional logic, i.e. NP-complete.

#### A. EXPSpace-complete fragments

In this subsection, we establish the following result.

**Theorem III.1.** *The TEL consistency problems for  $\text{THT}_2^2(\text{G})$ ,  $\text{THT}_2^1(\text{F}, \text{G})$ , and  $\text{THT}_2^2(\text{U})$  are all EXPSpace-hard.*

Theorem III.1 is proved by polynomial-time reductions from a domino-tiling problem for grids with rows of singly exponential length [22]. We fix an instance  $\mathcal{I}$  of such a problem, which is a tuple  $\mathcal{I} = \langle C, \Delta, n, d_{\text{init}}, d_{\text{final}} \rangle$ , where  $C$  is a finite set of colors,  $\Delta \subseteq C^4$  is a set of tuples  $\langle c_{\text{down}}, c_{\text{left}}, c_{\text{up}}, c_{\text{right}} \rangle$  of four colors, called *domino-types*,  $n > 0$  is a natural number (written in unary), and  $d_{\text{init}}, d_{\text{final}} \in \Delta$  are domino-types. A *tiling of  $\mathcal{I}$*  is a mapping  $f : [0, k] \times [0, 2^n - 1] \rightarrow \Delta$  for some  $k \geq 0$  satisfying the following:

- two adjacent cells in a row have the same color on the shared edge: for all  $(i, j) \in [0, k] \times [0, 2^n - 1]$  with  $j < 2^n - 1$ ,  $[f(i, j)]_{\text{right}} = [f(i, j + 1)]_{\text{left}}$ ;
- two adjacent cells in a column have the same color on the shared edge: for all  $(i, j) \in [0, k] \times [0, 2^n - 1]$  with  $i < k$ ,  $[f(i, j)]_{\text{up}} = [f(i + 1, j)]_{\text{down}}$ ;
- $f(0, 0) = d_{\text{init}}$  (*initialization*);
- $f(k, 2^n - 1) = d_{\text{final}}$  (*acceptance*).

**Remark III.1.** *Without loss of generality, we restrict ourselves to tilings  $f : [0, k] \times [0, 2^n - 1] \rightarrow \Delta$  of  $\mathcal{I}$  such that every cell except the last has content distinct from  $d_{\text{final}}$ , i.e. for all  $(i, j) \neq (k, 2^n - 1)$ ,  $f(i, j) \neq d_{\text{final}}$ .*

It is well-known that checking the existence of a tiling for  $\mathcal{I}$  is EXPSpace-complete [22]. In the following, for each  $\mathcal{L} \in \{\text{THT}_2^1(\text{F}, \text{G}), \text{THT}_2^2(\text{G}), \text{THT}_2^2(\text{U})\}$ , we construct in polynomial time an  $\mathcal{L}$ -formula which admits an equilibrium model iff there exists a tiling of  $\mathcal{I}$ . Hence, Theorem III.1 follows. We use the following set  $P$  of atomic propositions for encoding tilings of  $\mathcal{I}$ :

$$P = P_{\text{main}} \cup P_{\text{tag}} \cup \{u\} \quad P_{\text{tag}} = \{t_1, \dots, t_9\}$$

$$P_{\text{main}} = \Delta \cup \{\$ \} \cup P_{\text{num}} \quad P_{\text{num}} = [1, n] \times \{0, 1\}$$

The atomic propositions in  $P_{\text{num}} \subseteq P_{\text{main}}$  are used to encode the value of a  $n$ -bits counter numbering the cells of one row of a tiling. In particular, a cell with content  $d \in \Delta$  and column number  $j \in [0, 2^n - 1]$  is encoded by the words in

$$\{(1, b_1)\}^+ \dots \{(n, b_n)\}^+ \{d\}^+$$

where  $b_1 \dots b_n$  is the binary encoding of the column number  $j$ . Moreover, a row is encoded by words of the form  $\{\$ \}^h \cdot \text{cell}_0 \dots \text{cell}_{2^n - 1}$  for some  $h \geq 1$ , listing the encodings of cells from left to right. Thus, a tiling  $f$  is encoded by finite words  $w$  over  $2^{P_{\text{main}}}$  where  $w$  corresponds to a sequence of row encodings, starting from the first row of  $f$ . Note that for all  $i \geq 0$ ,  $w(i)$  contains exactly one atomic proposition in  $P_{\text{main}}$ . The extra symbols in  $P_{\text{tag}}$  and the additional proposition  $u$  are used to mark segments of infinite words  $\text{H}$  in order to check that the projection of  $\text{H}$  over  $P_{\text{main}}$  has no prefix which encodes a tiling.

*Reductions:* Here we focus on the fragment  $\text{THT}_2^1(\mathbf{F}, \mathbf{G})$ . The reductions for  $\text{THT}_2^2(\mathbf{G})$  and  $\text{THT}_2^2(\mathbf{U})$  are in Appendix VII-B. Our main tool is a notion of *pseudo-tiling code*.

**Definition III.1** (Pseudo-tiling codes for  $\text{THT}_2^1(\mathbf{F}, \mathbf{G})$ ). An interpretation  $(H, T)$  (over  $P$ ) is a pseudo-tiling code for  $\text{THT}_2^1(\mathbf{F}, \mathbf{G})$  if the following holds:

Unboundedness: for infinitely many  $i \geq 0$ ,  $u \in H(i)$ .

Pseudo-tiling T-requirement:  $\$ \in T(0)$  and:

- $T(i) \cap P_{\text{main}}$  is a singleton and  $T(i) \cap P_{\text{main}} = H(i) \cap P_{\text{main}}$  for all  $i \geq 0$ ;
- there is  $i \geq 0$  such that  $d_{\text{final}} \in T(i)$ .
- either for all  $i \geq 0$ ,  $u \in T(i)$  and  $T(i) \cap P_{\text{tag}} = P_{\text{tag}}$  (full requirement), or  $u \notin T(0)$  and for all  $i \geq 0$ ,  $T(i) \cap P_{\text{tag}}$  is a singleton.

H-requirement: if  $H \neq T$  (i.e.,  $H \sqsubset T$ ), then  $u \notin H(0)$  and  $H(i) \cap P_{\text{tag}}$  is a singleton for all  $i \geq 0$ .

A pseudo-tiling code  $(H, T)$  for  $\text{THT}_2^1(\mathbf{F}, \mathbf{G})$  is *good* if it satisfies the full requirement. We observe the following fact.

**Remark III.2.** Let  $(T, T)$  be a total pseudo-tiling code for  $\text{THT}_2^1(\mathbf{F}, \mathbf{G})$  which is not good. Then,  $u \notin T(0)$  and there exists  $H$  such that  $H \sqsubset T$ ,  $(H, T)$  is a pseudo-tiling code and for all  $i \geq 0$ ,  $H(i) \cap (P \setminus \{u\}) = T(i) \cap (P \setminus \{u\})$ .

We construct a  $\text{THT}_2^1(\mathbf{F}, \mathbf{G})$  formula  $\varphi_{\mathcal{I}}$  whose equilibrium models are the good pseudo-tiling codes  $(T, T)$  such that the projection of some prefix of  $T$  over  $P_{\text{main}}$  encodes a tiling. In particular, we ensure that for a good pseudo-tiling code  $(T, T)$ , there exists  $H \sqsubset T$  such that  $(H, T) \models \varphi_{\mathcal{I}}$  iff  $(H, T)$  is a pseudo-tiling code and  $H$  is a “slice” version of  $T$  witnessing that  $T$  has no prefix which encodes a tiling. The construction of  $\varphi_{\mathcal{I}}$  consists of three steps. First, we define a  $\text{THT}_2^1(\mathbf{F}, \mathbf{G})$  formula capturing the pseudo-tiling codes.

**Proposition III.1.** One can construct in polynomial time a  $\text{THT}_2^1(\mathbf{F}, \mathbf{G})$  formula  $\varphi_{\text{pseudo}}$  such that  $(H, T) \models \varphi_{\text{pseudo}}$  iff  $(H, T)$  is a pseudo-tiling code for  $\text{THT}_2^1(\mathbf{F}, \mathbf{G})$ .

*Proof:* The  $\text{THT}_2^1(\mathbf{F}, \mathbf{G})$  formula  $\varphi_{\text{pseudo}}$  is given by

$$\begin{aligned} & (\mathbf{G} \mathbf{F} u) \wedge \$ \wedge \mathbf{G} \left( \bigvee_{p \in P_{\text{main}}} (p \wedge \bigwedge_{p' \in P_{\text{main}} \setminus \{p\}} \neg p') \right) \wedge \\ & (\mathbf{F} d_{\text{final}}) \wedge \mathbf{G} \left( \bigvee_{p \in P_{\text{tag}}} p \right) \wedge \\ & \left( \left[ u \vee \mathbf{F} \left( \bigvee_{(p, p') \in P_{\text{tag}} : p \neq p'} (p \wedge p') \right) \right] \rightarrow \mathbf{G} \left( u \wedge \bigwedge_{p \in P_{\text{tag}}} p \right) \right) \end{aligned}$$

The first conjunct captures the unboundedness requirement, while the remaining conjuncts capture the pseudo-tiling T-requirement and the H-requirement. ■

Second, we use a family of  $\text{THT}_2^1(\mathbf{F}, \mathbf{G})$  formulas to mark by propositions in  $P_{\text{tag}}$  segments of infinite words on  $2^{P_{\text{main}}}$ .

**Proposition III.2.** Let  $t_{i_1}, \dots, t_{i_k}$  be distinct propositions in  $P_{\text{tag}}$ , and  $P_1, \dots, P_k$  be non-empty subsets of  $P_{\text{main}}$ . Then, one can construct in polynomial time a  $\text{THT}_2^1(\mathbf{F}, \mathbf{G})$  formula

$\theta(i_1|P_1, \dots, i_k|P_k)$  over  $P \setminus \{u\}$  such that: for all good pseudo-tiling codes  $(H, T)$  for  $\text{THT}_2^1(\mathbf{F}, \mathbf{G})$  with  $H \neq T$ ,

$$(H, T) \models \theta(i_1|P_1, \dots, i_k|P_k) \text{ iff}$$

the projection of  $H$  over  $P_{\text{tag}}$  is in  $\{t_{i_1}\}^+ \dots \{t_{i_{k-1}}\}^+ \{t_{i_k}\}^\omega$  and for all  $1 \leq j \leq k$ , all the main propositions which label the segment of  $H$  marked by  $t_{i_j}$  are in  $P_j$ . Moreover,

$$(H, T) \models \theta(i_1|P_1, \dots, i_k|P_k) \text{ iff } H \models_{\text{LTL}} \theta(i_1|P_1, \dots, i_k|P_k)$$

*Proof:* For a good pseudo-tiling code  $(H, T)$  with  $H \neq T$ ,  $T(i) \cap P_{\text{tag}} = P_{\text{tag}}$  and  $H(i) \cap P_{\text{tag}}$  is a singleton for all  $i \geq 0$ . Hence,  $(H, T) \not\models \psi$  and  $(T, T) \models \psi$ , where  $\psi = \bigwedge_{p \in P_{\text{tag}}} \mathbf{G} p$ .

Then,  $\theta(i_1|P_1, \dots, i_k|P_k)$  is given by

$$\begin{aligned} & \underbrace{\left( \bigvee_{t \in P_{\text{tag}} \setminus \{t_{i_1}, \dots, t_{i_k}\}} \mathbf{F} t \right) \rightarrow \psi}_{\text{H is only marked by tag propositions in } \{t_{i_1}, \dots, t_{i_k}\}} \wedge \\ & \underbrace{\bigwedge_{j \in [1, k]} \mathbf{F} t_{i_j}}_{\text{every tag } t_{i_j} \text{ marks some position of H}} \wedge \\ & \underbrace{\left( \bigvee_{j \in [1, k]} \bigvee_{p \in P_{\text{main}} \setminus P_j} \mathbf{F}(t_{i_j} \wedge p) \right) \rightarrow \psi}_{\text{each } t_{i_j}\text{-position in H is labeled by a main proposition in } P_j} \wedge \\ & \underbrace{\left( \bigvee_{r, s \in [1, k] : s < r} \mathbf{F}(t_{i_r} \wedge \mathbf{F} t_{i_s}) \right) \rightarrow \psi}_{\text{the tags } t_{i_j} \text{ mark H according to the order } t_{i_1}, \dots, t_{i_k}} \end{aligned}$$

The crucial step in the construction of  $\varphi_{\mathcal{I}}$  is represented by the following result.

**Proposition III.3.** One can construct in polynomial time a  $\text{THT}_2^1(\mathbf{F}, \mathbf{G})$  formula  $\varphi_{\text{bad}}$  over  $P \setminus \{u\}$  such that for all total interpretations  $(T, T)$  which are good pseudo-tiling codes for  $\text{THT}_2^1(\mathbf{F}, \mathbf{G})$ , there exists a good pseudo-tiling code for  $\text{THT}_2^1(\mathbf{F}, \mathbf{G})$  of the form  $(H, T)$  with  $H \neq T$  and satisfying  $\varphi_{\text{bad}}$  iff there is no prefix of  $T$  whose projection over  $P_{\text{main}}$  encodes a tiling. Moreover, for all good pseudo-tiling codes  $(H, T)$  for  $\text{THT}_2^1(\mathbf{F}, \mathbf{G})$  with  $H \neq T$ ,  $(H, T) \models \varphi_{\text{bad}}$  iff  $H \models_{\text{LTL}} \varphi_{\text{bad}}$ .

*Proof:* the  $\text{THT}_2^1(\mathbf{F}, \mathbf{G})$  formula  $\varphi_{\text{bad}}$  consists of various disjuncts which capture all the possible conditions such that for a total good pseudo-tiling code  $(T, T)$ , no prefix of  $T$  encodes a tiling iff some of these conditions is satisfied. These bad conditions can be summarized as follows, where for an LTL interpretation  $T$  over  $P$ , a prefix of  $T$  is *incomplete* if it has no position labeled by  $d_{\text{final}}$ :

- The content of the first cell is not  $d_{\text{init}}$ .
- Either some  $\$$ -position is preceded by an incomplete prefix and is followed by a  $\Delta$ -position, or some  $P_{\text{num}}$ -position is preceded by an incomplete prefix and is followed by a  $\$$ -position.
- No cell preceded by an incomplete prefix has content  $d_{\text{final}}$  and is the last cell of a row.

- There are segments in  $(\{\$\} \cup \Delta \setminus \{d_{final}\}) P_{num}^+ \Delta^+$ , preceded by incomplete prefixes, such that the suffix in  $P_{num}^+ \Delta^+$  is not a correct encoding of a cell.
- There is a row preceded by an incomplete prefix whose first (resp., last) cell has column number distinct from 0 (resp.,  $2^n - 1$ ).
- There are adjacent cells in a row, preceded by an incomplete prefix, whose column numbers are not consecutive.
- *Bad row (resp., column) condition*: there are two adjacent cells in a row (resp., column), preceded by an incomplete prefix, which have different color on the shared edge.

The above conditions are expressed in  $\text{THT}_2^1(\mathcal{F}, \mathcal{G})$  by exploiting the formulas  $\theta(i_1|P_1, \dots, i_k|P_k)$  of Proposition III.2. Here, we focus on the construction of the formula expressing the bad column condition (a full proof of Proposition III.3 is in Appendix VII-A). Such a formula is defined below, where we use the following short hands:  $R_m := P_{main} \setminus \{d_{final}\}$ ,  $R_c := P_{main} \setminus \{\$, d_{final}\}$ , and  $\Delta_R := \Delta \setminus \{d_{final}\}$ . Notice that we use the tag propositions  $t_2$  and  $t_7$  (resp.,  $t_3$  and  $t_8$ ) to mark the cell-numbers (resp., the contents) of two cells.

$$\left( \theta(1|R_m, 2|P_{num}, 3|\Delta_R, 4|R_c, 5|\{\$\}, 6|R_c, 7|P_{num}, 8|\Delta, 9|P_{main}) \right. \\ \left. \vee \theta(1|R_m, 2|P_{num}, 3|\Delta_R, 4|R_c, 5|\{\$\}, 7|P_{num}, 8|\Delta, 9|P_{main}) \vee \right. \\ \left. \theta(1|R_{main}, 2|P_{num}, 3|\Delta_R, 5|\{\$\}, 6|R_c, 7|P_{num}, 8|\Delta, 9|P_{main}) \right)$$

mark with  $t_2$  and  $t_7$  the cell-numbers of two cells  $c$  and  $c'$  of two adjacent rows

$$\bigwedge_{i \in [1, n]} \bigvee_{b \in \{0, 1\}} \left( F((i, b) \wedge t_2) \wedge F((i, b) \wedge t_7) \right) \wedge \\ \underbrace{\bigvee_{(d, d') \in \Delta \times \Delta: d_{up} \neq (d')_{down}} \left( F(d \wedge t_3) \wedge F(d' \wedge t_8) \right)}_{\text{the marked cells } c \text{ and } c' \text{ do not have the same color on the shared edge}}$$

By using Propositions III.1 and III.3, we deduce the following result from which Theorem III.1 for the fragment  $\text{THT}_2^1(\mathcal{F}, \mathcal{G})$  directly follows.

**Lemma III.1.** *One can construct in polynomial time a  $\text{THT}_2^1(\mathcal{F}, \mathcal{G})$  formula  $\varphi_{\mathcal{I}}$  such that there is an equilibrium model of  $\varphi_{\mathcal{I}}$  iff there is a tiling of  $\mathcal{I}$ .*

*Proof:* Let  $\varphi_{pseudo}$  and  $\varphi_{bad}$  be the  $\text{THT}_2^1(\mathcal{F}, \mathcal{G})$  formulas of Propositions III.1 and III.3, respectively. Then:

$$\varphi_{\mathcal{I}} = \varphi_{pseudo} \wedge (u \vee \varphi_{bad})$$

We now prove that the construction is correct. First, assume that there exists an equilibrium model  $(\mathcal{T}, \mathcal{T})$  of  $\varphi_{\mathcal{I}}$ . By construction of  $\varphi_{\mathcal{I}}$  and Proposition III.1,  $(\mathcal{T}, \mathcal{T})$  is a pseudo-tiling code. We claim that  $(\mathcal{T}, \mathcal{T})$  is good as well. We assume the contrary and derive a contradiction. By Remark III.2,  $u \notin \mathcal{T}(0)$  and there exists  $\mathcal{H} \sqsubset \mathcal{T}$  such that  $(\mathcal{H}, \mathcal{T})$  is a pseudo-tiling code and for all  $i \geq 0$ ,  $\mathcal{H}(i) \cap (P \setminus \{u\}) = \mathcal{T}(i) \cap (P \setminus \{u\})$ . Since  $u \notin \mathcal{T}(0)$  and  $(\mathcal{T}, \mathcal{T}) \models \varphi_{\mathcal{I}}$ ,  $(\mathcal{T}, \mathcal{T}) \models \varphi_{bad}$ . Moreover, since  $\varphi_{bad}$  is a formula over  $P \setminus \{u\}$  (Proposition III.3), by

Proposition III.1 we obtain that  $(\mathcal{H}, \mathcal{T})$  satisfies  $\varphi_{\mathcal{I}}$ , which contradicts the hypothesis that  $(\mathcal{T}, \mathcal{T})$  is an equilibrium model of  $\varphi_{\mathcal{I}}$ . Thus,  $(\mathcal{T}, \mathcal{T})$  is a good pseudo-tiling code. If no prefix of  $\mathcal{T}$  encodes a tiling, by Proposition III.3 there exists  $\mathcal{H} \sqsubset \mathcal{T}$  such that  $(\mathcal{H}, \mathcal{T}) \models \varphi_{bad}$  and  $(\mathcal{H}, \mathcal{T})$  is a pseudo-tiling code; hence, by Proposition III.1,  $(\mathcal{H}, \mathcal{T})$  satisfies  $\varphi_{\mathcal{I}}$ , which contradicts the assumption that  $(\mathcal{T}, \mathcal{T})$  is an equilibrium model. Thus, some prefix of  $\mathcal{T}$  encodes a tiling. Hence, there exists a tiling of  $\mathcal{I}$ .

Now, assume that there exists a tiling  $f$  of  $\mathcal{I}$ . Let  $(\mathcal{T}, \mathcal{T})$  be any *good* pseudo-tiling code such that the projection of some prefix of  $\mathcal{T}$  over  $P_{main}$  is an encoding of  $f$ . Note that such a  $(\mathcal{T}, \mathcal{T})$  exists. Since  $u \in \mathcal{T}(0)$ , by construction and Proposition III.1,  $(\mathcal{T}, \mathcal{T})$  satisfies  $\varphi_{\mathcal{I}}$ . We assume that  $(\mathcal{T}, \mathcal{T})$  is not an equilibrium model and derive a contradiction, hence, the result follows. Thus, there is  $\mathcal{H} \sqsubset \mathcal{T}$  such that  $(\mathcal{H}, \mathcal{T}) \models \varphi_{\mathcal{I}}$ . By construction and Proposition III.1,  $(\mathcal{H}, \mathcal{T})$  is a pseudo-tiling code. Moreover, since  $(\mathcal{T}, \mathcal{T})$  is good,  $(\mathcal{H}, \mathcal{T})$  is good as well. Since  $\mathcal{H} \neq \mathcal{T}$ ,  $u \notin \mathcal{H}(0)$  (Definition III.1). Hence, being  $(\mathcal{H}, \mathcal{T}) \models \varphi_{\mathcal{I}}$ , by construction,  $(\mathcal{H}, \mathcal{T}) \models \varphi_{bad}$ . By Proposition III.3 there is no prefix of  $\mathcal{T}$  which encodes a tiling. This contradicts the hypothesis, and we are done. ■

## B. The fragment $\text{THT}_1$

We establish that the TEL consistency problem for the simple fragment  $\text{THT}_1$ , where no nesting of temporal modalities is allowed, is already NEXPTIME-complete even for the smaller fragments  $\text{THT}_1(\mathcal{F}, \mathcal{G})$ ,  $\text{THT}_1(\mathcal{U})$ , and  $\text{THT}_1(\mathcal{R})$ .

### 1) Lower Bounds:

**Theorem III.2.** *The TEL consistency problems for  $\text{THT}_1^2(\mathcal{F}, \mathcal{G})$ ,  $\text{THT}_1^2(\mathcal{U})$ , and  $\text{THT}_1^2(\mathcal{R})$  are NEXPTIME-hard.*

Theorem III.2 is proved by polynomial-time reductions from a domino-tiling problem for grids with rows and columns of exponential length [23]. An instance  $\mathcal{I} = \langle C, \Delta, n, d_{init}, d_{final} \rangle$  of this problem is as in the proof of Theorem III.1. However, here, a tiling of  $\mathcal{I}$  is defined as a mapping  $f : [0, 2^n - 1] \times [0, 2^n - 1] \rightarrow \Delta$ , i.e., the number of rows and the number of columns is  $2^n$ . It is well-known that checking the existence of a tiling for  $\mathcal{I}$  is NEXPTIME-complete [23]. We focus on the fragment  $\text{THT}_1^2(\mathcal{F}, \mathcal{G})$ . The reductions for the fragments  $\text{THT}_1^2(\mathcal{U})$  and  $\text{THT}_1^2(\mathcal{R})$  are given in Appendix VII-D.

*Encoding of tilings for  $\text{THT}_1^2(\mathcal{F}, \mathcal{G})$ :* we use the following set  $P$  of propositions:

$$P = P_{main} \cup P_{tag} \cup \{u\} \quad P_{main} = \Delta \cup P_{num}^r \cup P_{num}^c$$

$$P_{num}^r = \{r\} \times [1, n] \times \{0, 1\} \quad P_{num}^c = \{c\} \times [1, n] \times \{0, 1\}$$

$$P_{tag} = \{t_1, t_2, t_3\} \times \{\bar{p} \mid p \in P_{num}^r \cup P_{num}^c\}$$

We use the atomic propositions in  $P_{num}^r$  (resp.,  $P_{num}^c$ ) to encode the value of a  $n$ -bits counter numbering the  $2^n$  rows (resp., columns) of a tiling. In particular, a cell with content  $d \in \Delta$ , row number  $i \in [0, 2^n - 1]$ , and column number  $j \in [0, 2^n - 1]$  is encoded by the subset of  $P_{main}$  given by

$$\{d, (r, 1, b_1), \dots, (r, n, b_n), (c, 1, b'_1), \dots, (c, n, b'_n)\}$$

where  $b_1 \dots b_n$  (resp.,  $b'_1, \dots, b'_n$ ) is the binary encoding of the row number  $i$  (resp., column number  $j$ ). We call such subsets of  $P_{main}$  *cell-codes*. A tiling  $f$  is then encoded by the infinite words  $w$  over  $2^{P_{main}}$  satisfying the following:

- for all  $i, j \in [0, 2^n - 1]$ , there is  $h \geq 0$  such that  $w(h)$  is the cell-code of the  $(i, j)^{th}$  cell of  $f$ ;
- for all  $h \geq 0$ ,  $w(h)$  encodes the  $(i, j)^{th}$  cell of  $f$  for some  $i, j \in [0, 2^n - 1]$ .

The extra symbols in  $P_{tag}$  and the additional proposition  $u$  are used to mark infinite words  $H$  in order to check that the projection of  $H$  over  $P_{main}$  does not encode a tiling. In particular, a *cell-number code* is a subset of  $P_{tag}$  of the form

$$\{(\overline{r, 1, b_1}), \dots, (\overline{r, n, b_n}), (\overline{c, 1, b'_1}), \dots, (\overline{c, n, b'_n})\}$$

*Reduction for  $THT_1^2(F, G)$ :* as in the proof of Theorem III.1, we use a notion of *pseudo-tiling code*.

**Definition III.2** (Pseudo-tiling codes for  $THT_1^2(F, G)$ ). *An interpretation  $(H, T)$  (over  $P$ ) is a pseudo-tiling code for  $THT_1^2(F, G)$  if the following holds:*

Pseudo-tiling T-requirement: *for all  $i \geq 0$ ,  $T(i) \cap P_{main}$  is a cell-code and  $H(i) \cap P_{main} = T(i) \cap P_{main}$ . Moreover,*

- *there is  $i \geq 0$  such that  $T(i) \cap P_{main}$  has row-number 0, column-number 0 and  $d_{init} \in T(i)$  (initialization);*
- *there is  $i \geq 0$  such that  $T(i) \cap P_{main}$  has row-number  $2^n - 1$ , column-number  $2^n - 1$ , and  $d_{final} \in T(i)$  (acceptance).*

Full T-requirement: *for all  $i$ ,  $T(i) \cap P_{tag} = P_{tag}$  and  $u \in T(i)$ ;*

H-requirement: *if  $H \neq T$ , then  $u \notin H(i)$  for all  $i \geq 0$ , and:*

- *either there is a cell-number code  $P' \subseteq P_{tag}$  such that the projection of  $H$  over  $P_{tag}$  is  $(P')^\omega$ ;*
- *or for all  $i \geq 0$ ,  $H(i) \cap P_{tag}$  is a singleton contained in  $\{t_1, t_2, t_3\}$ .*

We construct in polynomial time a  $THT_1^2(F, G)$  formula  $\varphi_{\mathcal{I}}$  in such a way that (i) the total interpretations captured by  $\varphi_{\mathcal{I}}$  are the total interpretations  $(T, T)$  which are pseudo-tiling codes for  $THT_1^2(F, G)$ , and (ii) there exists  $H \sqsubset T$  such that  $(H, T) \models \varphi_{\mathcal{I}}$  iff the projection of  $T$  over  $P_{main}$  does not encode a tiling. The construction of  $\varphi_{\mathcal{I}}$  consists of two steps. First, we define a formula capturing the pseudo-tiling codes.

**Proposition III.4.** *One can construct in polynomial time a  $THT_1^2(F, G)$  formula  $\varphi_{pseudo}$  such that  $(H, T) \models \varphi_{pseudo}$  iff  $(H, T)$  is a pseudo-tiling code for  $THT_1^2(F, G)$ .*

The proof of Proposition III.4 is crucially based on the use of nested implication. In particular, we exploit the conjunct  $\neg u \rightarrow u$  which is satisfied by an interpretation  $(H, T)$  iff  $u \in T(0)$ . For details, see Appendix VII-C. The second step in the construction of  $\varphi_{\mathcal{I}}$  is given by the following result.

**Proposition III.5.** *One can construct in polynomial time a  $THT_1^2(F, G)$  formula  $\varphi_{bad}$  such that for all total interpretations  $(T, T)$  which are pseudo-tiling codes for  $THT_1^2(F, G)$ , there exists a pseudo-tiling code for  $THT_1^2(F, G)$  of the form  $(H, T)$  with  $H \neq T$  and satisfying  $\varphi_{bad}$  iff the projection of  $T$  over  $P_{main}$  does not encode a tiling.*

*Proof:* First, for all  $t, t' \in \{t_1, t_2, t_3\}$  and  $\tau \in \{r, c\}$ , we consider the  $THT_1^1(F, G)$  formula  $\varphi(t, t', \tau)$  given by

$$\{ \bigvee_{i \in [1, n]} [F((t \vee t') \wedge (\tau, i, 0)) \wedge F((t \vee t') \wedge (\tau, i, 1))] \} \rightarrow u$$

Evidently, for each pseudo-tiling code  $(H, T)$  for  $THT_1^2(F, G)$  with  $H \neq T$ ,  $(H, T) \models \varphi(t, t', r)$  (resp.,  $(H, T) \models \varphi(t, t', c)$ ) iff for all the positions of  $H$  marked by the propositions  $t$  and  $t'$ , the associated cell-codes have the same row-number (resp., column-number). Then the  $THT_1^1(F, G)$  formula  $\varphi_{bad}$  consist of four disjuncts. The first disjunct checks that there is a cell-number  $(i, j)$  such that no cell-code has cell-number  $(i, j)$ .

$$\left( \bigvee_{p \in P_{tag} \setminus \{t_1, t_2, t_3\}} Fp \right) \wedge$$

all the positions of  $H$  are marked by the same cell-number code  $P' \subseteq P_{tag}$

$$G \left( \bigvee_{i \in [1, n]} \bigvee_{\tau \in \{r, c\}} \bigvee_{b \in \{0, 1\}} [(\tau, i, b) \wedge (\tau, i, 1 - b)] \right)$$

at every position, the current cell-code has cell-number non-corresponding to  $P'$

The second disjunct checks that there are two cell-codes with the same cell-number but distinct content.

$$(Ft_1) \wedge \bigwedge_{\tau \in \{r, c\}} \varphi(t_1, t_1, \tau) \wedge \bigvee_{d, d' \in \Delta: d \neq d'} [F(t_1 \wedge d) \wedge F(t_1 \wedge d')]$$

Finally, the third (resp., fourth) disjunct checks that there are two adjacent cells in a column (resp., row) which do *not* have the same color on the shared edge. We illustrate the construction of the fourth disjunct.

$$\underbrace{(Ft_1) \wedge (Ft_2) \wedge \varphi(t_1, t_1, r) \wedge \varphi(t_2, t_2, r) \wedge \varphi(t_1, t_2, c)}_{\text{mark two cells } cl_1 \text{ and } cl_2 \text{ with the same column number}} \wedge$$

$$\bigvee_{i \in [1, n]} \left[ F((r, i, 0) \wedge t_1) \wedge F((r, i, 1) \wedge t_2) \wedge \right.$$

$$\bigwedge_{j \in [1, i-1]} \left( F((r, j, 1) \wedge t_1) \wedge F((r, j, 0) \wedge t_2) \right) \wedge$$

$$\bigwedge_{j \in [i+1, n]} \bigvee_{b \in \{0, 1\}} \left( F((r, j, b) \wedge t_1) \wedge F((r, j, b) \wedge t_2) \right) \left. \right] \wedge$$

$cl_1$  and  $cl_2$  have consecutive row-numbers

$$\bigvee_{(d, d') \in \Delta \times \Delta: d_{up} \neq (d')_{down}} [F(t_1 \wedge d) \wedge F(t_2 \wedge d')]$$

the cells  $cl_1$  and  $cl_2$  do not have the same color on the shared edge

By construction, for all pseudo-tiling codes  $(H, T)$  for  $THT_1^2(F, G)$  such that  $H \neq T$ , if  $(H, T) \models \varphi_{bad}$  then  $T$  does not encode a tiling. On the other hand, for each total pseudo-tiling code  $(T, T)$  for  $THT_1^2(F, G)$  such that  $T$  does not encode a tiling, there exists a pseudo-tiling code for  $THT_1^2(F, G)$  of the form  $(H, T)$  such that  $H \neq T$  and  $(H, T)$  satisfies  $\varphi_{bad}$ . Hence, Proposition III.5 follows. ■

The  $THT_1^2(F, G)$  formula  $\varphi_{\mathcal{I}}$  is defined as follows:

$$\varphi_{\mathcal{I}} = \varphi_{pseudo} \wedge (u \vee \varphi_{bad})$$

where  $\varphi_{pseudo}$  and  $\varphi_{bad}$  are the  $THT_1^2(F, G)$  formulas of Proposition III.4 and III.5, respectively. By Propositions III.4

and III.5, we easily deduce the following result, hence, Theorem III.2 for the fragment  $\text{THT}_1^2(\mathcal{F}, \mathcal{G})$  directly follows.

**Lemma III.2** (Correctness of the construction). *There is an equilibrium model of  $\varphi_{\mathcal{I}}$  iff there is a tiling of  $\mathcal{I}$ .*

2) *Upper Bound for  $\text{CON}(\text{THT}_1)$ :* An interpretation  $M$  is *strongly ultimately periodic* if there is  $i \geq 0$  such that  $M(k) = M(i)$  for all  $k \geq i$ . In such a case, the size of  $M$  is defined as  $j + 1$ , where  $j$  is the smallest  $i$  satisfying the previous condition. In order to solve  $\text{CON}(\text{THT}_1)$ , we first show that we can restrict ourselves to the equilibrium models which are strongly ultimately periodic and whose sizes are singly exponential in the size of the given formula.

**Lemma III.3.** *Let  $\varphi$  be a  $\text{THT}_1$  formula having some equilibrium model. Then, there exists a strongly ultimately periodic equilibrium model of  $\varphi$  of size at most  $2 + 2^{|\varphi|}$ .*

The proof of Lemma III.3, which is detailed in Appendix VII-E, exploits a notion of bisimilarity and contraction for interpretations. Bisimilar interpretations are indistinguishable from  $\text{THT}_1$  formulas, and the notion of contraction, which ensures bisimilarity, allows to ‘extract’ from a total interpretation a strongly ultimately periodic interpretation of size singly exponential in  $|\varphi|$  by preserving the property of being an equilibrium model of a  $\text{THT}_1$  formula.

Next, we show that for a  $\text{THT}_1$  formula  $\varphi$  and a strongly ultimately periodic total interpretation  $M$  of size singly exponential in  $|\varphi|$ , checking that  $M$  is an equilibrium model of  $\varphi$  can be done in time singly exponential in  $|\varphi|$ . For this, we use a notion of extracted interpretation depending on  $\varphi$ , which generalizes a similar notion exploited in [24] for solving LTL satisfiability for  $\text{THT}_1$  (considered as LTL fragment).

**Definition III.3** (Witness Extraction). Given  $\varphi \in \text{THT}_1$  and an interpretation  $M = (H, T)$ , a *witness pattern of  $M$  for  $\varphi$*  is an infinite sequence  $n_0 < n_1 < \dots$  of increasing natural numbers such that there is  $k \geq 0$  so that  $M(n_i) = M(n_{k+1})$  for all  $i \geq k+1$ , and the finite set of positions  $W = \{n_0, \dots, n_k\}$  *minimally* satisfies the following conditions:

- $0 \in W$  and if there is some subformula of  $\varphi$  of the form  $X\psi$ , then  $1 \in W$ ;
- if  $M$  is not total, then for some  $i$ ,  $H(i) \subset T(i)$  and  $i \in W$ ;
- for each subformula  $\varphi_1 \mathbf{U} \varphi_2$  of  $\varphi$ :
  - if  $M \models \varphi_1 \mathbf{U} \varphi_2$ , then the smallest position  $i$  such that  $M, i \models \varphi_2$  is in  $W$ .
  - if  $M \not\models \varphi_1 \mathbf{U} \varphi_2$  and  $M \models F\varphi_2$ , then the smallest position  $i$  such that  $M, i \not\models \varphi_1$  is in  $W$ .
- for each subformula  $\varphi_1 \mathbf{R} \varphi_2$  of  $\varphi$ :
  - if  $M \not\models \varphi_1 \mathbf{R} \varphi_2$ , then the smallest position  $i$  such that  $M, i \not\models \varphi_2$  is in  $W$ .
  - if  $M \models \varphi_1 \mathbf{R} \varphi_2$  and  $M \not\models G\varphi_2$ , then the smallest position  $i$  such that  $M, i \models \varphi_1 \wedge \varphi_2$  is in  $W$ .

Note that witness patterns of  $M$  for  $\varphi$  exist. A *witness extraction of  $M$  for  $\varphi$*  is an interpretation  $M_W$  of the form  $M_W = M(n_0), M(n_1), \dots$ , where  $n_0 < n_1 < \dots$  is a witness pattern of  $M$  for  $\varphi$ . Evidently,  $M_W$  is strongly ultimately

periodic with size at most  $|\varphi| + 3$ .

We establish the following result whose proof is in Appendix VII-F.

**Lemma III.4.** *Given  $\varphi \in \text{THT}_1$ , the following holds.*

- 1) *Let  $M$  and  $M'$  be two interpretations such that  $M' = M(n_0), M(n_1), \dots$  where  $n_0 < n_1 < \dots$  is an infinite sequence of increasing natural numbers containing all the positions of some witness pattern of  $M$  for  $\varphi$ . Then, for each subformula  $\psi$  of  $\varphi$ ,  $M \models \psi$  iff  $M' \models \psi$ .*
- 2) *Let  $M = (T, T)$  be a total strongly ultimately periodic interpretation satisfying  $\varphi$  of size  $m$ . Then  $M$  is an equilibrium model of  $\varphi$  iff for each  $H \sqsubset T$  such that  $(H, T)$  is a strongly ultimately periodic interpretation of size at most  $m + |\varphi| + 3$ ,  $(H, T) \not\models \varphi$ .*

By Lemmata III.3 and III.4, we obtain the desired result.

**Theorem III.3.**  *$\text{CON}(\text{THT}_1)$  is in NEXPTIME.*

*Proof:* Let  $\varphi$  be a  $\text{THT}_1$  formula. By Lemma III.3, if  $\varphi$  has an equilibrium model, then there is some equilibrium model  $(T, T)$  of  $\varphi$  which is strongly ultimately periodic and whose size is at most  $2 + 2^{|\varphi|}$ . Nondeterministically guessing such a  $(T, T)$  and checking that  $(T, T)$  satisfies  $\varphi$  can be done in singly exponential time. Moreover, by Lemma III.4, for verifying that  $(T, T)$  is an equilibrium model, it suffices to check that for every strongly ultimately periodic interpretation  $(H_W, T_W)$  of size at most  $|\varphi| + 3$ , it holds that  $(H_W, T_W) \not\models \varphi$  whenever  $(H_W, T_W)$  satisfies the following condition.

*Downward condition:* there is  $H \sqsubset T$  such that  $(H, T)$  is strongly ultimately periodic with size at most  $5 + 2^{|\varphi|} + |\varphi|$ , and  $(H_W, T_W)$  is a witness extraction of  $(H, T)$  for  $\varphi$ .

By Definition III.3, one can deduce that checking whether  $(H_W, T_W)$  satisfies the downward condition can be done in singly exponential (deterministic) time. Thus, since the number of strongly ultimately periodic interpretations of size at most  $|\varphi| + 3$  is singly exponential in the size of  $|\varphi|$ , membership in NEXPTIME for  $\text{CON}(\text{THT}_1)$  follows. ■

#### IV. TRACTABLE FRAGMENTS

We now turn to the syntactical fragments of  $\text{THT}$ , as defined in Subsection II-A, which are not captured by the results of Section III. For each of these fragments, except the fragment  $\text{THT}^0$ , we will show that the TEL consistency problem is complete for some complexity class in  $\{\text{NP}, \Sigma_2, \text{PSPACE}\}$ . For the fragment  $\text{THT}^0$ , where no use of implication (and negation) is allowed, we are only able to provide a PSPACE lower bound, as established by the following theorem. Notice that Theorem IV.1, whose proof is given in Appendix VIII-A, is, in fact, surprising since a  $\text{THT}^0$  formula is always satisfiable.

**Theorem IV.1.**  *$\text{CON}(\text{THT}^0)$  is PSPACE-hard.*

##### A. The fragment $\text{THT}_1(\mathcal{X}, \mathcal{G})$

The proposed approach for the fragment  $\text{THT}_1(\mathcal{X}, \mathcal{G})$  is based on the notion of witness extraction of Definition III.3. The main result is as follows.



**Lemma IV.1.** Let  $\varphi \in \text{THT}_1(X, G)$  and  $M$  be an equilibrium model of  $\varphi$ . Then, every witness extraction of  $M$  for  $\varphi$  is still an equilibrium model of  $\varphi$ .

*Proof:* let  $M = (T, T)$  be an equilibrium model of  $\varphi$  and  $M_W = (T_W, T_W)$  be a witness extraction of  $M$  for  $\varphi$ . We show that  $M_W$  is an equilibrium model of  $\varphi$ . By Lemma III.4(1),  $M_W$  satisfies  $\varphi$ . Fix  $H_W \sqsubset T_W$ . It remains to prove that  $(H_W, T_W) \not\models \varphi$ . Let  $n_0 < n_1 < \dots$  be the witness pattern of  $M$  for  $\varphi$  such that  $T_W = T(n_0), T(n_1), \dots$ . Define  $H$  as the LTL interpretation where: for all  $i \geq 0$ , if  $i = n_j$  for some  $j$ , then  $H(i) = H_W(j)$ ; otherwise,  $H(i) = T(i)$ . Evidently,  $H \sqsubset T$ . Let  $M' = (H, T)$  and  $M'_W = (H_W, T_W)$ . Note that  $M'_W = M'(n_0), M'(n_1), \dots$ . We prove that for each subformula  $\psi$  of  $\varphi$ ,  $M' \models \psi$  iff  $M'_W \models \psi$ . Hence, since  $M' \not\models \varphi$  ( $(T, T)$  is an equilibrium model of  $\varphi$ ), the result follows.

The unique non-trivial case is when  $\psi = G\psi'$ . The implication  $M'_W \not\models G\psi' \Rightarrow M' \not\models G\psi'$  easily follows from the construction and the fact that  $\psi'$  has no temporal modalities. Now, assume that  $M'_W \models G\psi'$ . We need to prove that for all  $i \geq 0$ ,  $M', i \models \psi'$ . If  $i = n_j$  for some  $j \geq 0$ , then  $M'(i) = M'_W(j)$ . Thus, since  $\psi'$  has no temporal modalities, by hypothesis, the result follows. Otherwise, by construction,  $M'(i) = (T(i), T(i))$ . We assume that  $M', i \not\models \psi'$  and derive a contradiction. Since  $\psi'$  has no temporal modalities, we obtain that  $(T, T) \not\models G\psi'$ . By Lemma III.4(1),  $(T_W, T_W) \not\models G\psi'$ , hence,  $M'_W = (H_W, T_W) \not\models G\psi'$  as well (Proposition II.1(1)), which contradicts the hypothesis, and we are done. ■

By applying Lemmata III.4 and IV.1, we obtain:

**Theorem IV.2.**  $\text{CON}(\text{THT}_1(X, G))$  is  $\Sigma_2$ -complete.

*Proof:* The lower bound directly follows from  $\Sigma_2$ -completeness of  $\text{CON}(\text{HT})$  [5], [21]. For the matching upper bound, let  $\varphi$  be a  $\text{THT}_1(X, G)$  formula. By Lemma IV.1 and Definition III.3, if  $\varphi$  has an equilibrium model, then there is some equilibrium model  $(T, T)$  of  $\varphi$  which is strongly ultimately periodic and whose size is at most  $|\varphi| + 3$ . Nondeterministically guessing such a  $(T, T)$  and checking that  $(T, T)$  satisfies  $\varphi$  can be done in polynomial time. Moreover, by Lemma III.4(2), to verify that  $(T, T)$  is an equilibrium model, it suffices to check that each strongly ultimately periodic interpretation of size at most  $2(|\varphi| + 3)$  and of the form  $(H, T)$  such that  $H \sqsubset T$ , does not satisfy  $\varphi$ . Universally guessing such a  $(H, T)$  and checking that it does not satisfy  $\varphi$  can be done in polynomial time. Hence, the result follows. ■

### B. The fragments $\text{THT}^1(X, R)$ , $\text{THT}^1(X, U)$ , and $\text{THT}_1^1$

By Theorem III.1, the TEL consistency problem for  $\text{THT}^1$  where there is no nesting of implication is already EXPSpace-complete. However, we now show that for the relevant fragments  $\text{THT}^1(X, R)$  and  $\text{THT}^1(X, U)$  of  $\text{THT}^1$ , where the combined use of modalities  $U$  and  $R$  is disallowed, the problem is instead PSPACE-complete. Additionally, we establish that  $\text{CON}(\text{THT}_1^1)$  is NP-complete.

1) *The fragments  $\text{THT}^1(X, R)$  and  $\text{THT}_1^1$ :* For these two fragments, we first show that LTL satisfiability always guarantees the existence of minimal LTL models.

**Theorem IV.3.** Every LTL satisfiable  $\text{THT}^1(X, R)$  (resp.,  $\text{THT}_1^1$ ) formula admits a minimal LTL model.

*Proof:* We focus on the fragment  $\text{THT}^1(X, R)$  (for the fragment  $\text{THT}_1^1$ , details can be found in Appendix VIII-B). The proof for  $\text{THT}^1(X, R)$  is by contradiction. So, assume that there exists a  $\text{THT}^1(X, R)$  formula  $\varphi$  such that  $\varphi$  is LTL satisfiable but there is no minimal LTL model of  $\varphi$ . Let  $(T_n)_{n \geq 0}$  be any infinite sequence of LTL models of  $\varphi$  satisfying the following:

- $T_0$  is any LTL model of  $\varphi$ ;
- for all  $n \geq 0$ ,  $T_{n+1}$  is any LTL model of  $\varphi$  such that  $T_{n+1} \sqsubset T_n$  and the following holds:  
*Finite minimal requirement for  $n$ :* there is no LTL model  $H$  of  $\varphi$  such that  $H \sqsubset T_n$  and: (i) for all  $i \in [0, n+1]$ ,  $H(i) \subseteq T_{n+1}(i)$ , and (ii) for some  $i \in [0, n+1]$ ,  $H(i) \subset T_{n+1}(i)$ .

By hypothesis, such a sequence  $(T_n)_{n \geq 0}$  exists. Let  $T$  be the LTL interpretation defined as follows: for all  $i \geq 0$ ,

$$T(i) := \bigcap_{n \geq 0} T_n(i)$$

We will show that  $T$  is a minimal model of  $\varphi$ , which contradicts the assumption. Hence, the result follows. First, we observe the following.

*Claim 1:* 1)  $T_{n+1} \sqsubset T_n$  and  $T \sqsubset T_n$  for all  $n \geq 0$ ;

2) for all  $i \geq 0$ , there is  $k \geq 0$  such that for all  $n \geq k$ ,  $T_n(i) = T(i)$ ;

3) for all  $H \sqsubset T$ ,  $H \not\models_{\text{LTL}} \varphi$ .

*Proof of Claim 1:* Properties 1 and 2 directly follow by construction. For Property 3, let  $H \sqsubset T$ , and  $n$  be any natural number such that for some  $i \in [0, n+1]$ ,  $H(i) \subset T(i)$ . By Property 1,  $T \sqsubset T_{n+1}$  and  $T_{n+1} \sqsubset T_n$ . Hence,  $H \sqsubset T_n$  and: (i) for all  $i \in [0, n+1]$ ,  $H(i) \subseteq T_{n+1}(i)$ , and (ii) for some  $i \in [0, n+1]$ ,  $H(i) \subset T_{n+1}(i)$ . Thus, by the finite minimal requirement for  $n$ ,  $H \not\models_{\text{LTL}} \varphi$ . □

Next, we prove the following.

*Claim 2:* Let  $\phi$  be a  $\text{THT}^1(X, R)$  formula and  $i \geq 0$  such that  $T, i \models_{\text{LTL}} \neg\phi$ . Then, there is  $k \geq 0$  such that for all  $n \geq k$ ,  $T_n, i \models_{\text{LTL}} \neg\phi$ .

*Proof of Claim 2:* first, we recall that for a  $\text{THT}$  formula  $\psi$  (considered as LTL formula), the *LTL normal form* of  $\psi$  is obtained by pushing inward negations to propositional literals using De Morgan's laws, the duality between  $U$  and  $R$ , and the fact that in the classical interpretation of implication, formula  $\xi_1 \rightarrow \xi_2$  can be rewritten as  $\neg\xi_1 \vee \xi_2$ . If  $\psi'$  is the LTL normal form of  $\psi$ , then  $\psi$  and  $\psi'$  are globally equivalent, i.e., for all LTL interpretations  $T$  and positions  $i \geq 0$ ,  $T, i \models_{\text{LTL}} \psi$  iff  $T, i \models_{\text{LTL}} \psi'$ .

Now, we prove Claim 2. Let  $\phi$  be a  $\text{THT}^1(X, R)$  formula and  $i \geq 0$  such that  $T, i \models \neg\phi$ . The proof is by induction on the structure of the normal form  $\psi$  of  $\neg\phi$ . We crucially use the following fact: since  $\phi \in \text{THT}^1(X, R)$ , every subformula of  $\psi$  of the form  $\psi_1 R \psi_2$  is *positive*, i.e.  $\psi_1 R \psi_2 \in \text{THT}^0$ .

- $\psi = p$  or  $\psi = \neg p$  for some  $p \in P$ : the result directly follows from Claim 1(2).
- $\psi = \psi_1 \vee \psi_2$  or  $\psi = \psi_1 \wedge \psi_2$ : the result easily follows from the induction hypothesis.
- $\psi = X\psi_1$ : we apply the induction hypothesis on  $\psi_1$  and position  $i + 1$ .
- $\psi = \psi_1 U \psi_2$ : hence, there exists  $j \geq i$  such that  $T, j \models_{\text{LTL}} \psi_2$  and  $T, m \models_{\text{LTL}} \psi_1$  for all  $m \in [i, j - 1]$ . By applying the induction hypothesis, there exist  $k_i, \dots, k_j$  such that  $T_n, j \models_{\text{LTL}} \psi_2$  for all  $n \geq k_j$ , and for all  $m \in [i, j - 1]$  and  $n \geq k_m$ ,  $T_n, m \models_{\text{LTL}} \psi_1$ . Thus, by taking  $k = \max(\{k_i, \dots, k_j\})$ , the result follows.
- $\psi = \psi_1 R \psi_2$ : hence,  $\psi_1 R \psi_2$  is a positive formula, i.e.,  $\psi_1 R \psi_2 \in \text{THT}^0$ . Evidently, for all LTL interpretations  $H$  and  $H'$  such that  $H \sqsubseteq H'$  and for all positive formulas  $\xi$ ,  $H, i \models_{\text{LTL}} \xi$  implies  $H', i \models_{\text{LTL}} \xi$ . Thus, since  $T \sqsubseteq T_n$  for all  $n \geq 0$ , the result follows.  $\square$

Since  $T_n$  is an LTL model of  $\varphi$  for all  $n \geq 0$ , by Claim 2, we deduce that  $T \models_{\text{LTL}} \varphi$ . Thus, by Claim 1(3),  $T$  is a minimal LTL model of  $\varphi$  which concludes.  $\blacksquare$

We establish now the main results for  $\text{THT}^1(X, R)$  and  $\text{THT}_1^1$ .

**Theorem IV.4.** *A  $\text{THT}^1(X, R)$  (resp.,  $\text{THT}_1^1$ ) formula  $\varphi$  has an equilibrium model iff  $\varphi$  is LTL satisfiable. Moreover,  $\text{CON}(\text{THT}^1(X, R))$  and  $\text{CON}(\text{THT}_1^1(R))$  are PSPACE-complete, while  $\text{CON}(\text{THT}_1^1)$  is NP-complete.*

*Proof:* For the first part of Theorem IV.4, if  $\varphi$  has an equilibrium model, then by Proposition II.1(3),  $\varphi$  is LTL satisfiable. For the converse direction, assume that  $\varphi$  is LTL satisfiable. By Theorem IV.3,  $\varphi$  has a minimal LTL model  $T$ . Since  $\varphi \in \text{THT}^1$ , by Proposition II.1(3-4),  $(T, T)$  is an equilibrium model of  $\varphi$ .

By well-known lower bounds for LTL [16], [24], LTL-satisfiability for the fragment  $\text{THT}^1(R)$  is PSPACE-hard. Thus, since LTL satisfiability is PSPACE-complete, and LTL satisfiability for the fragment  $\text{THT}_1^1$  is NP-complete [24], the second part of Theorem IV.4 follows as well.  $\blacksquare$

2) *The fragment  $\text{THT}^1(X, U)$ :* For this fragment, we show that the TEL consistency problem can be reduced in linear-time to LTL-satisfiability.

Given an interpretation  $(H, T)$  and a position  $i \geq 0$ ,  $i$  is an *empty position* of  $(H, T)$  if  $H(i) = \emptyset$ . A total interpretation having a finite number of non-empty positions is said to be *almost-empty*. A THT formula  $\varphi$  satisfies the *almost-empty requirement* if every temporal equilibrium of  $\varphi$  is almost-empty. We first observe the following.

**Lemma IV.2.** *Let  $\varphi \in \text{THT}^1$  and satisfy the almost-empty requirement. Then, there exists an equilibrium model of  $\varphi$  iff the following formula is LTL-satisfiable*

$$\varphi \wedge \text{FG} \bigwedge_{p \in P} \neg p \quad (1)$$

*Proof:* Let  $(T, T)$  be an equilibrium model of  $\varphi$ . Since  $\varphi$  satisfies the almost-empty requirement, by Proposition II.1(3),

$T$  is an LTL model of formula (1). Now, assume that formula (1) has an LTL-model. Hence, there is an almost-empty interpretation  $(T, T)$  such that  $T \models_{\text{LTL}} \varphi$  and  $(T, T) \models \varphi$ . Since the number of non-empty positions of  $T$  is finite, we can also assume that for all  $H \sqsubset T$ ,  $H \not\models_{\text{LTL}} \varphi$  (i.e.,  $T$  is a minimal LTL model of  $\varphi$ ). Since  $\varphi \in \text{THT}^1$ , by Proposition II.1(4), there is no  $H \sqsubset T$  such that  $(H, T) \models \varphi$ . Thus,  $(T, T)$  is an equilibrium model of  $\varphi$ , which concludes.  $\blacksquare$

Next, we establish that the formulas in the fragment  $\text{THT}(X, U)$  satisfy the almost-empty requirement. For this, we need additional definitions. For a THT formula  $\varphi$ ,  $d_X(\varphi)$  denotes the nesting depth of modality  $X$  in  $\varphi$ .

**Definition IV.1** (Set of witnesses for  $\text{THT}(X, U)$ ). Let  $\varphi$  be a  $\text{THT}(X, U)$  formula and  $M = (T, T)$  be a total interpretation. We denote by  $\text{Fin}(\varphi, M)$  (resp.,  $\text{Inf}(\varphi, M)$ ) the set of subformulas  $\psi_1 U \psi_2$  of  $\varphi$  such that the number of positions  $i$  so that  $M, i \models \psi_2$  is finite and non-empty (resp., infinite). Note that  $\text{Fin}(\varphi, M) \cap \text{Inf}(\varphi, M) = \emptyset$ . For  $\psi_1 U \psi_2 \in \text{Fin}(\varphi, M) \cup \text{Inf}(\varphi, M)$ , a *witness of  $M$  for  $\psi_1 U \psi_2$*  is a position  $j$  such that  $M, j \models \psi_2$ .

Let  $\text{Fin}(\varphi, M) = \{\phi_1, \dots, \phi_k\}$ . Fix an ordering  $\xi_1, \dots, \xi_m$  of the subformulas in  $\text{Inf}(\varphi, M)$  such that for all  $i, j \in [1, m]$ , if  $i \neq j$  and  $\xi_i$  is a subformula of  $\xi_j$ , then  $i > j$ . A *set of witnesses of  $M$  for  $\varphi$*  is any set of the form

$$\{(0, \varphi), (j_1, \phi_1), \dots, (j_k, \phi_k)\} \cup \{(h_1, \xi_1), \dots, (h_m, \xi_m)\}$$

such that the following holds, where  $\ell = \max(\{j_1, \dots, j_k\})$ :

- $j_i$  is the the greatest witness of  $M$  for  $\phi_i$  for all  $i \in [1, k]$ ;
- $h_j$  is a witness of  $M$  for  $\xi_j$  for all  $j \in [1, m]$ ;
- $h_1 > \ell + d_X(\varphi)$  and  $h_{j+1} > h_j + d_X(\varphi)$  for all  $j \in [1, m - 1]$ .

Note that by definition of  $\text{Inf}(\varphi, M)$ , sets of witnesses of  $M$  for  $\varphi$  exist. Moreover, such sets have cardinality at most  $|\varphi| + 1$ .

**Lemma IV.3.** *Let  $\varphi$  be a  $\text{THT}(X, U)$  formula and  $M$  be an equilibrium model of  $\varphi$ . Then,  $M$  is almost-empty.*

*Proof:* Fix a set of witnesses  $W$  of  $M = (T, T)$  for  $\varphi$ . Let  $\ell$  be the greatest position occurring in  $W$ . We define an LTL interpretation  $H_W \sqsubseteq T$  as follows:

- for all  $i \geq 0$ ,  $H_W(i) = T(i)$  if  $i \leq \ell + d_X(\varphi)$ , and  $H_W(i) = \emptyset$  otherwise.

We show that  $H_W = T$ , hence,  $M = (T, T)$  is almost empty, and the result follows. For this, since  $M = (T, T)$  is an equilibrium model of  $\varphi$ , it suffices to prove that  $(H_W, T), 0 \models \varphi$ . Since  $(0, \varphi) \in W$ , the result directly follows from the following claim, which can be proved by structural induction on  $\psi$  by using Definition IV.1 and Proposition II.1(1). For details, see Appendix VIII-C.

*Claim:* let  $(j, \psi) \in W$  and  $\xi$  be a subformula of  $\psi$ . Then:

- 1) for all  $k \in [0, d_X(\psi)]$  such that  $d_X(\xi) \leq d_X(\psi) - k$ ,  $(T, T), j + k \models \xi$  iff  $(H_W, T), j + k \models \xi$ .
- 2) for all  $k \in [0, j]$ ,  $(T, T), k \models \xi$  iff  $(H_W, T), k \models \xi$ .  $\blacksquare$

By well-known lower bounds for LTL [16], [24], LTL-satisfiability of formulas of the form  $\varphi \wedge \text{FG} \bigwedge_{p \in P} \neg p$ , where

$\varphi$  is a  $\text{THT}^1(\mathbf{U})$  formula is PSPACE-hard. Thus, since LTL-satisfiability is PSPACE-complete, by Lemmata IV.2 and IV.3, we obtain the following result.

**Corollary IV.1.** *The TEL consistency problems for  $\text{THT}^1(\mathbf{X}, \mathbf{U})$  and  $\text{THT}^1(\mathbf{U})$  are PSPACE-complete.*

### C. The fragment $\text{THT}(\mathbf{X}, \mathbf{F})$

It is well-known that LTL-satisfiability for the LTL fragment corresponding to  $\text{THT}(\mathbf{X}, \mathbf{F})$  is already PSPACE-complete [16]. By contrast and surprisingly, we show that the TEL consistency problem for  $\text{THT}(\mathbf{X}, \mathbf{F})$  is just  $\Sigma_2$ -complete.

The size of an almost-empty total interpretation  $(\mathbf{T}, \mathbf{T})$  is  $h + 1$  where  $h$  is the smallest position such that  $\mathbf{T}(i) = \emptyset$  for all  $i \geq h$ . The main result for  $\text{THT}(\mathbf{X}, \mathbf{F})$  is as follows.

**Proposition IV.1.** *Let  $\varphi$  be a  $\text{THT}(\mathbf{X}, \mathbf{F})$  formula. If  $\varphi$  has an equilibrium model, then  $\varphi$  has an almost-empty equilibrium model of size at most  $|\varphi|^3$ .*

Given a  $\text{THT}(\mathbf{X}, \mathbf{F})$  formula  $\varphi$ , nondeterministically guessing an almost-empty total interpretation  $(\mathbf{T}, \mathbf{T})$  of size at most  $|\varphi|^3$  and checking that  $(\mathbf{T}, \mathbf{T})$  satisfies  $\varphi$  can be done in polynomial time. Moreover, universally guessing  $\mathbf{H} \sqsubset \mathbf{T}$  and checking that  $(\mathbf{H}, \mathbf{T})$  does not satisfy  $\varphi$  can be done in polynomial time. Hence, since  $\text{CON}(\text{HT})$  is  $\Sigma_2$ -complete, by Proposition IV.1, we obtain the following.

**Corollary IV.2.** *The TEL consistency problems for  $\text{THT}(\mathbf{X}, \mathbf{F})$ ,  $\text{THT}(\mathbf{X})$ , and  $\text{THT}(\mathbf{F})$  are  $\Sigma_2$ -complete.*

We now proceed with the proof of Proposition IV.1 which consists of the following two Lemmata IV.4 and IV.5.

**Lemma IV.4.** *Let  $\varphi$  be a  $\text{THT}(\mathbf{X}, \mathbf{F})$  formula and  $\mathbf{M} = (\mathbf{T}, \mathbf{T})$  be an equilibrium model of  $\varphi$ . Then,  $\mathbf{M}$  has at most  $d_X(\varphi) \cdot (|\varphi| + 1)$  non-empty positions.*

*Proof:* Let  $W$  be a set of witnesses of  $\mathbf{M}$  for  $\varphi$  according to Definition IV.1. By Definition IV.1,  $W$  has cardinality at most  $|\varphi| + 1$ . Now, we define an LTL interpretation  $\mathbf{H}_W \sqsubseteq \mathbf{T}$  as follows: for all  $i \geq 0$ , if there is  $(j, \psi) \in W$  such that  $j \leq i$  and  $i - j \leq d_X(\varphi)$ , then  $\mathbf{H}_W(i) = \mathbf{T}(i)$ ; otherwise,  $\mathbf{H}_W(i) = \emptyset$ .

By construction, the set of non-empty positions of the interpretation  $(\mathbf{H}_W, \mathbf{T})$  has cardinality at most  $d_X(\varphi) \cdot (|\varphi| + 1)$ . We show that  $\mathbf{H}_W = \mathbf{T}$ , hence, the result follows. For this, since  $\mathbf{M} = (\mathbf{T}, \mathbf{T})$  is an equilibrium model of  $\varphi$ , it suffices to prove that  $(\mathbf{H}_W, \mathbf{T}), 0 \models \varphi$ . Since  $(0, \varphi) \in W$ , the result directly follows from the following claim, whose proof, based on Definition IV.1 and Proposition II.1(1), is given in Appendix VIII-D:

*Claim:* for all  $(i, \psi) \in W$ ,  $k \in [0, d_X(\psi)]$ , and subformulas  $\xi$  of  $\psi$  such that  $d_X(\xi) \leq d_X(\psi) - k$ ,  $(\mathbf{T}, \mathbf{T}), i + k \models \xi$  iff  $(\mathbf{H}_W, \mathbf{T}), i + k \models \xi$ . ■

The following result is straightforward (for details, see Appendix VIII-E).

**Lemma IV.5.** *Let  $\varphi$  be a  $\text{THT}(\mathbf{X}, \mathbf{F})$  formula,  $n \geq 1$ , and  $\mathbf{M} = (\mathbf{T}, \mathbf{T})$  be an equilibrium model of  $\varphi$  having  $n$  non-empty*

*positions. Then, there exists an almost-empty equilibrium model of  $\varphi$  of size at most  $n \cdot (d_X(\varphi) + 1)$ .*

## V. MINIMAL LTL SATISFIABILITY

In this section we establish the complexity of the *minimal LTL satisfiability problem*, i.e., checking for a given THT formula  $\varphi$ , whether  $\varphi$  has a minimal LTL model.

**Theorem V.1.** *Minimal LTL satisfiability is EXPSpace-complete even for the syntactical fragment  $\text{THT}_2^1(\mathbf{F}, \mathbf{G})$ .*

*Proof:* For the lower bound, let  $\mathcal{I}$  be an instance of the domino tiling problem considered in the proof of Theorem III.1, and  $\varphi_{\mathcal{I}}$  be the  $\text{THT}_2^1(\mathbf{F}, \mathbf{G})$  formula of Lemma III.1. We show that  $\varphi_{\mathcal{I}}$  has a minimal LTL model iff  $\varphi_{\mathcal{I}}$  has an equilibrium model. Hence, by Lemma III.1, the lower bound of Theorem V.1 follows. Since  $\varphi_{\mathcal{I}}$  is a  $\text{THT}_2^1(\mathbf{F}, \mathbf{G})$  formula, if  $\varphi_{\mathcal{I}}$  has a minimal LTL model  $\mathbf{T}$ , then by Proposition II.1(3–4),  $(\mathbf{T}, \mathbf{T})$  is an equilibrium model of  $\varphi_{\mathcal{I}}$ . For the converse implication, let  $(\mathbf{T}, \mathbf{T})$  be an equilibrium model of  $\varphi_{\mathcal{I}}$ . We assume that  $\mathbf{T}$  is not a minimal LTL model of  $\varphi_{\mathcal{I}}$  and derive a contradiction. Hence, by Proposition II.1(3), there is  $\mathbf{H} \sqsubset \mathbf{T}$  such that  $\mathbf{H} \models_{\text{LTL}} \varphi_{\mathcal{I}}$  and  $(\mathbf{H}, \mathbf{H}) \models \varphi_{\mathcal{I}}$ . By the proof of Lemma III.1,

$$\varphi_{\mathcal{I}} = \varphi_{\text{pseudo}} \wedge (u \vee \varphi_{\text{bad}})$$

where  $\varphi_{\text{pseudo}}$  and  $\varphi_{\text{bad}}$  are the  $\text{THT}_2^1(\mathbf{F}, \mathbf{G})$  formulas of Propositions III.1 and III.3, respectively. Moreover,  $(\mathbf{T}, \mathbf{T})$  is a good pseudo-tiling code for  $\text{THT}_2^1(\mathbf{F}, \mathbf{G})$ . Since  $(\mathbf{H}, \mathbf{H}) \models \varphi_{\mathcal{I}}$  and  $\mathbf{H} \sqsubset \mathbf{T}$ , by Proposition III.1 and Definition III.1, it follows that  $(\mathbf{H}, \mathbf{T})$  is a good pseudo-tiling code for  $\text{THT}_2^1(\mathbf{F}, \mathbf{G})$  and  $u \notin \mathbf{H}(0)$ . Thus, since  $\mathbf{H} \models_{\text{LTL}} \varphi_{\mathcal{I}}$ , we have that  $\mathbf{H} \models_{\text{LTL}} \varphi_{\text{bad}}$ . By Propositions III.1 and III.3, we obtain that  $(\mathbf{H}, \mathbf{T})$  satisfies  $\varphi_{\mathcal{I}}$ . This contradicts the assumption that  $(\mathbf{T}, \mathbf{T})$  is an equilibrium model of  $\varphi_{\mathcal{I}}$ , and we are done.

For the upper bound, we exploit an automata-theoretic approach. Let  $\varphi$  be a THT formula. It is well-known [25] that one can construct in singly exponential time a Büchi nondeterministic finite-state automaton (Büchi NFA)  $\mathcal{A}_{\varphi}$  over  $2^P$  whose accepted language  $\mathcal{L}(\mathcal{A})$  is the set of LTL interpretations which are LTL models of  $\varphi$ . Moreover, given a Büchi NFA  $\mathcal{A}$  over  $2^P$ , it is straightforward to construct in quadratic time a Büchi NFA  $K(\mathcal{A})$  such that  $\mathbf{T} \in \mathcal{L}(K(\mathcal{A}))$  iff  $\mathbf{T} \in \mathcal{L}(\mathcal{A})$  and there is  $\mathbf{H} \sqsubset \mathbf{T}$  such that  $\mathbf{H} \in \mathcal{L}(\mathcal{A})$ . Hence,  $K(\mathcal{A}_{\varphi})$  accepts the set of LTL models of  $\varphi$  which are not minimal. It follows that  $\varphi$  has a minimal LTL model iff

$$\mathcal{L}(\mathcal{A}_{\varphi}) \cap [(2^P)^{\omega} \setminus \mathcal{L}(K(\mathcal{A}_{\varphi}))] \neq \emptyset \quad (2)$$

Now, checking non-emptiness of Büchi NFA can be done in NLogspace. Moreover, the  $\omega$ -languages recognized by Büchi NFA are closed under intersection and complementation, and complementation involves a singly exponential blow-up. Thus, by well-known results [25]–[27], checking equation (2) can be done in single exponential space, which concludes the proof. ■

It is well-known that for both the considered standard version of LTL, whose interpretations are infinite words, and

finitary LTL (i.e. LTL interpreted over *finite* words), satisfiability is PSPACE-complete [16]. On the other hand, Theorem V.1 highlights a meaningful difference arising from interpreting LTL over finite words or infinite words. Indeed, while for finitary LTL, minimal satisfiability evidently coincides with satisfiability, for infinite words, minimal satisfiability turns out to be singly exponentially harder than satisfiability.

## VI. CONCLUSION

We conclude with some observations and future research directions. We have provided a systematic study of the computational complexity of the TEL consistency problem by considering natural syntactical fragments of THT. Our complexity results show that there is no difference in tractability between implication height 2 and  $k$  with  $k > 2$ , and the same holds for the temporal height. Moreover, unlike in the case of LTL, in THT dual temporal modalities need to be considered independently from one another, and they have quite different computational costs. Additionally, we have shown that minimal LTL satisfiability has, in the general case, the same complexity as checking TEL consistency. However, for some of the considered fragments, we have a different scenario. An example is the fragment  $\text{THT}_1$  where there is no nesting of temporal modalities: in this restricted case, the TEL consistency problem is NEXPTIME-complete, while, by Theorem IV.3, minimal LTL satisfiability coincides with LTL satisfiability, the latter being just NP-complete for the fragment  $\text{THT}_1$  [24].

Another subclass of THT formulas, called temporal logic programs (TLP) has been considered in [28], [29]. TLP conforms to a logic programming style and corresponds to a fragment of  $\text{THT}_2^2(X, F, G)$ . As shown in [28], for the TEL consistency problem, the general case reduces in polynomial time to the case of TLP formulas. Thus, our results imply that checking TEL consistency for TLP is already EXSPACE-complete.<sup>2</sup>

As future research, we aim to address expressiveness issues for the TEL framework. In particular, since we have individuated some non-trivial tractable fragments such as  $\text{THT}_1(X, G)$  and  $\text{THT}(X, F)$ , it would be interesting to study what kind of temporal reasoning problems they can express. Moreover, an important question is to investigate from a semantical point of view the considered syntactical hierarchy of THT fragments: is this hierarchy also semantically strict with respect to THT and/or TEL semantics? Another relevant issue is to provide alternative characterizations of the class of TEL languages (the  $\omega$ -languages of equilibrium models of THT formulas). It is known that this class is regular [2]. An intriguing open question is whether TEL languages are LTL-expressible.

## ACKNOWLEDGMENTS

Work supported by the projects VIVAC (TIN2012-38137-C02, Bozzelli) and MERLOT (TIN2013-42149-P, Pearce).

<sup>2</sup>In [29] TLP rules are divided into four different syntactical fragments. *Initial rules* are in  $\text{THT}_1^2(X)$ ; *fulfillment rules* are of two types, either in  $\text{THT}_2^2(G)$  or in  $\text{THT}_2^2(F, G)$ ; while so-called *dynamic rules* fall in the fragment  $\text{THT}_2^2(X, G)$ .

## REFERENCES

- [1] P. Cabalar and G. Vega, "Temporal equilibrium logic: A first approach," in *Proc. 11th EUROCAST*, ser. LNCS 4739. Springer, 2007, pp. 241–248.
- [2] P. Cabalar and S. Demri, "Automata-Based Computation of Temporal Equilibrium Models," in *Proc. 21st LOPSTR*, ser. LNCS 7225. Springer, 2011, pp. 57–72.
- [3] G. Brewka, T. Eiter, and M. Truszczynski, "Answer set programming at a glance," *Commun. ACM*, vol. 54, no. 12, pp. 92–103, 2011.
- [4] D. Pearce, "A new logical characterisation of stable models and answer sets," in *Proc. of NMELP'96*, ser. LNCS 1216. Springer, 1996, pp. 57–70.
- [5] —, "Equilibrium logic," *Ann. Math. Artif. Intell.*, vol. 47, no. 1-2, pp. 3–41, 2006.
- [6] A. Heyting, "Die formalen Regeln der intuitionistischen Logik," in three parts, *Sitzungsberichte der preussischen Akademie der Wissenschaften*. English translation of Part I in Mancosu 1998: 311–327.
- [7] D. Pearce and A. Valverde, "Towards a first order equilibrium logic for nonmonotonic reasoning," in *Proc. JELIA 2004*, Springer LNCS 3229, 2004, pp. 147–160.
- [8] P. Cabalar, L. F. del Cerro, D. Pearce, and A. Valverde, "A free logic for stable models with partial intensional functions," in *Proc. JELIA 2014*, Springer LNCS 8761, 2014, pp. 340–354.
- [9] J. de Bruijn, D. Pearce, A. Polleres, and A. Valverde, "Quantified equilibrium logic and hybrid rules," in *Proc. Web Reasoning and Rule Systems, RR 2007*, Springer LNCS 4524, 2007, pp. 58–72.
- [10] M. Fink and D. Pearce, "A logical semantics for description logic programs," in *Proc. JELIA 2010*, Springer LNCS 6341, 2010, pp. 156–168.
- [11] J. McCarthy and P. J. Hayes, "Some philosophical problems from the standpoint of artificial intelligence," in *Machine Intelligence*. Edinburgh University Press, 1969, pp. 463–502.
- [12] H. A. Kautz, "The logic of persistence," in *Proc. 5th AAAI*, 1986, pp. 401–405.
- [13] A. Pnueli, "The temporal logic of programs," in *Proc. 18th FOCS*. IEEE Computer Society, 1977, pp. 46–57.
- [14] P. Cabalar and M. Diéguez, "Strong equivalence of non-monotonic temporal theories," in *Proc. 14th KR*. AAAI Press, 2014.
- [15] —, "Stelp - A tool for temporal answer set programming," in *Proc. 11th LPNMR*, ser. LNCS 6645. Springer, 2011, pp. 370–375.
- [16] A. Sistla and E. Clarke, "The Complexity of Propositional Linear Temporal Logics," *J. ACM*, vol. 32, no. 3, pp. 733–749, 1985.
- [17] R. Fagin, J. Halpern, and M. Vardi, *Reasoning about knowledge*. MIT press Cambridge, 1995, vol. 4.
- [18] L. Giordano, A. Martelli, and D. T. Dupré, "Reasoning about actions with temporal answer sets," *TPLP*, vol. 13, no. 2, pp. 201–225, 2013.
- [19] F. Aguado, G. Pérez, and C. Vidal, "Integrating temporal extensions of answer set programming," in *Proc. 12th LPNMR*, ser. LNCS 8148. Springer, 2013, pp. 23–35.
- [20] P. Ferraris, "Answer Sets for Propositional Theories," in *Proc. 8th LPNMR*, ser. LNCS 3662. Springer, 2005, pp. 119–131.
- [21] T. Eiter and G. Gottlob, "On the Computational Cost of Disjunctive Logic Programming: Propositional Case," *Ann. Math. Artif. Intell.*, vol. 15, no. 3-4, pp. 289–323, 1995.
- [22] D. Harel, *Algorithmics: The Spirit of Computing*. Wesley, 2nd edition, 1992.
- [23] P. V. E. Boas, "The Convenience of Tilings," in *In Complexity, Logic, and Recursion Theory*. Marcel Dekker Inc, 1997, pp. 331–363.
- [24] S. Demri and P. Schnoebelen, "The Complexity of Propositional Linear Temporal Logics in Simple Cases," *Inf. Comput.*, vol. 174, no. 1, pp. 84–103, 2002.
- [25] M. Y. Vardi and P. Wolper, "Reasoning about infinite computations," *Inf. Comput.*, vol. 115, no. 1, pp. 1–37, 1994.
- [26] A. P. Sistla, M. Y. Vardi, and P. Wolper, "The complementation problem for Büchi automata with applications to temporal logic," *Theor. Comput. Sci.*, vol. 49, pp. 217–237, 1987.
- [27] S. Safra, "Complexity of automata on infinite objects," 1989, PhD thesis, Weizmann Institute of Science, Rehovot, 1989.
- [28] P. Cabalar, "A normal form for linear temporal equilibrium logic," in *Proc. 12th JELIA*, ser. LNCS 6341. Springer, 2010, pp. 64–76.
- [29] F. Aguado, P. Cabalar, M. Diéguez, G. Pérez, and C. Vidal, "Temporal equilibrium logic: a survey," *Journal of Applied Non-Classical Logics*, vol. 23, no. 1-2, pp. 2–24, 2013.

# Appendix

## VII. PROOFS FROM SECTION III

### A. Full proof of Proposition III.3

**Proposition III.3.** *One can construct in polynomial time a  $THT_2^1(F, G)$  formula  $\varphi_{bad}$  over  $P \setminus \{u\}$  such that for all total interpretations  $(T, T)$  which are good pseudo-tiling codes for  $THT_2^1(F, G)$ , there exists a good pseudo-tiling code for  $THT_2^1(F, G)$  of the form  $(H, T)$  with  $H \neq T$  and satisfying  $\varphi_{bad}$  iff there is no prefix of  $T$  whose projection over  $P_{main}$  encodes a tiling. Moreover, for all good pseudo-tiling codes  $(H, T)$  for  $THT_2^1(F, G)$  with  $H \neq T$ ,  $(H, T) \models \varphi_{bad}$  iff  $H \models_{LTL} \varphi_{bad}$ .*

*Proof:* In the proof, we use the following short-hands:

$$\begin{aligned} R_{main} &= P_{main} \setminus \{d_{final}\} \\ P_{cell} &= P_{main} \setminus \{\$ \} \\ R_{cell} &= P_{cell} \setminus \{d_{final}\} \\ \Delta_R &= \Delta \setminus \{d_{final}\} \end{aligned}$$

Moreover, for an LTL interpretation  $T$  over  $P$ , we say that a prefix of  $T$  is *incomplete* if it has no position labeled by  $d_{final}$ .

The  $THT_2^1(F, G)$  formula  $\varphi_{bad}$  is defined as follows:

$$\begin{aligned} \varphi_{bad} = & \varphi_{bad\_in} \vee \varphi_{bad\_ord} \vee \varphi_{bad\_acc} \vee \varphi_{bad\_cell} \vee \varphi_{bad\_first} \vee \varphi_{bad\_last} \vee \\ & \varphi_{bad\_inc} \vee \varphi_{bad\_rr} \vee \varphi_{bad\_cr} \end{aligned}$$

where for a total good pseudo-tiling code  $(T, T)$ , the different disjuncts in the definition of  $\varphi_{bad}$  capture all the possible conditions such that no prefix of  $T$  encodes a tiling iff some of these conditions is satisfied. The construction of such disjuncts exploits the formulas  $\theta(i_1|P_1, \dots, i_k|P_k)$  of Proposition III.2.

The disjunct  $\varphi_{bad\_in}$  checks that the content of the first cell is not  $d_{init}$ .

$$\varphi_{bad\_in} = \theta(1|\{\$, 2|P_{num}, 3|\Delta \setminus \{d_{init}\}, 4|P_{main})$$

The disjunct  $\varphi_{bad\_ord}$  is used to check that *either* some  $\$$ -position is preceded by an incomplete prefix and is followed by a  $\Delta$ -position, *or* some  $P_{num}$ -position is preceded by an incomplete prefix and is followed by a  $\$$ -position.

$$\begin{aligned} \varphi_{bad\_ord} = & \theta(1|R_{main}, 2|\{\$, 3|\Delta, 4|P_{main}) \vee \theta(2|\{\$, 3|\Delta, 4|P_{main}) \vee \\ & \theta(1|R_{main}, 2|P_{num}, 3|\{\$, 4|P_{main}) \end{aligned}$$

The disjunct  $\varphi_{bad\_acc}$  asserts that there is *no* cell  $c$  preceded by an incomplete prefix such that  $c$  has content  $d_{final}$  and  $c$  is the last cell of a row (recall that for a pseudo-tiling code some position is labeled by  $d_{final}$ ).

$$\begin{aligned} \varphi_{bad\_acc} = & \bigvee_{i=1}^{i=n} \left( \theta(1|R_{main}, 2|\{(i, 0)\}, 3|P_{num}, 4|\{d_{final}\}, 5|P_{main}) \vee \right. \\ & \left. \theta(1|R_{main}, 2|\{(i, 0)\}, 4|\{d_{final}\}, 5|P_{main}) \right) \end{aligned}$$

The disjunct  $\varphi_{bad\_cell}$  is used to individuate segments in  $(\Delta_R \cup \{\$\})P_{num}^+\Delta^+$  which are preceded by incomplete prefixes and such that the suffix in  $P_{num}^+\Delta^+$  is not a correct encoding of a cell.

$$\begin{aligned}
\varphi_{bad\_cell} = & \underbrace{\bigvee_{(d,d') \in (\Delta \setminus \{d_{final}\}) \times \Delta: d \neq d'} \theta(1|R_{main}, 2|\{d\}, 3|\{d'\}, 4|P_{main})}_{\text{a cell contains two distinct elements in } \Delta} \vee \\
& \bigvee_{i=1}^{i=n} \bigvee_{(b,b') \in \{0,1\}: b \neq b'} \left( \theta(1|R_{main}, 2|\{(i,b)\}, 3|\{(i,b')\}, 4|P_{main}) \vee \right. \\
& \quad \left. \theta(1|R_{main}, 2|\{(i,b)\}, 3|P_{num}, 4|\{(i,b')\}, 5|P_{main}) \right) \vee \\
& \underbrace{\hspace{10em}}_{\text{two distinct bits in the encoding of a cell have the same bit position}} \\
& \bigvee_{((i,b),(j,b')) \in P_{num} \times P_{num}: i > j} \left( \theta(1|R_{main}, 2|\{(i,b)\}, 3|\{(j,b')\}, 4|P_{main}) \vee \right. \\
& \quad \left. \theta(1|R_{main}, 2|\{(i,b)\}, 3|P_{num}, 4|\{(j,b')\}, 5|P_{main}) \right) \vee \\
& \underbrace{\hspace{10em}}_{\text{the bit positions in the encoding of a cell are not ordered correctly}} \\
& \bigvee_{i=1}^{i=n} \left( \theta(1|R_{main}, 2|\Delta_R \cup \{\$\}, 3|P_{num} \setminus \{i\} \times \{0,1\}, 4|\Delta, 5|P_{main}) \vee \right. \\
& \quad \left. \theta(1|\{\$\}, 3|P_{num} \setminus \{i\} \times \{0,1\}, 4|\Delta, 5|P_{main}) \right) \vee \\
& \underbrace{\hspace{10em}}_{\text{some bit position in the encoding of a cell is absent}}
\end{aligned}$$

The disjunct  $\varphi_{bad\_first}$  (resp.,  $\varphi_{bad\_last}$ ) checks the existence of rows which are preceded by incomplete prefixes and whose first (resp., last) cell has column number distinct from 0 (resp.,  $2^n - 1$ ).

$$\begin{aligned}
\varphi_{bad\_first} = & \bigvee_{i=1}^{i=n} \left( \theta(1|R_{main}, 2|\{\$\}, 3|P_{num}, 4|\{(i,1)\}, 5|P_{main}) \vee \right. \\
& \quad \theta(1|R_{main}, 2|\{\$\}, 4|\{(i,1)\}, 5|P_{main}) \vee \\
& \quad \left. \theta(2|\{\$\}, 3|P_{num}, 4|\{(i,1)\}, 5|P_{main}) \vee \theta(2|\{\$\}, 4|\{(i,1)\}, 5|P_{main}) \right) \\
\\
\varphi_{bad\_last} = & \theta(1|R_{main}, 2|\{(n,0)\}, 3|\Delta, 4|\{\$\}, 5|P_{main}) \vee \\
& \bigvee_{i=1}^{i=n} \theta(1|R_{main}, 2|\{(i,0)\}, 3|P_{num}, 4|\Delta, 5|\{\$\}, 6|P_{main})
\end{aligned}$$

The disjunct  $\varphi_{bad\_inc}$  selects adjacent cells in a row whose column numbers are not consecutive; moreover, the rightmost cell is preceded by an incomplete prefix.

$$\begin{aligned} \varphi_{bad\_inc} = & \left( \theta(1|R_{main}, 2|\Delta_R \cup \{\$\}, 3|P_{num}, 4|\Delta_R, 5|P_{num}, 6|\Delta, 7|P_{main}) \vee \right. \\ & \left. \underbrace{\theta(1|\{\$\}, 3|P_{num}, 4|\Delta_R, 5|P_{num}, 6|\Delta, 7|P_{main})}_{\text{mark by } t_3 \text{ and } t_5 \text{ the cell-numbers of two adjacent cells in a row}} \right) \wedge \\ & \psi_{bad\_inc} \end{aligned}$$

where  $\psi_{bad\_inc}$  asserts that the cell numbers marked by  $t_3$  and  $t_5$ , respectively, are not consecutive.

$$\begin{aligned} \psi_{bad\_inc} = & \bigwedge_{i=1}^{i=n} \left( F((i, 1) \wedge t_3) \wedge F((i, 0) \wedge t_5) \right) \vee \bigvee_{i=1}^{i=n} \left( F((i, 0) \wedge t_3) \wedge F((i, 1) \wedge t_5) \wedge \right. \\ & \left. \left[ \bigvee_{j=1}^{j=i-1} (F((j, 0) \wedge t_3) \wedge F((j, 1) \wedge t_5)) \vee \bigvee_{j=i+1}^{j=n} \bigvee_{b, b' \in \{0,1\}: b \neq b'} (F((j, b) \wedge t_3) \wedge F((j, b') \wedge t_5)) \right] \right) \end{aligned}$$

The disjunct  $\varphi_{bad\_rr}$  checks that there are two adjacent cells in a row which do *not* have the same color on the shared edge; moreover, the rightmost cell is preceded by an incomplete prefix.

$$\varphi_{bad\_rr} = \bigvee_{(d, d') \in (\Delta \setminus \{d_{final}\}) \times \Delta: d_{right} \neq (d')_{left}} \theta(1|R_{main}, 2|\{d\}, 3|P_{num}, 4|\{d'\}, 5|P_{main})$$

Finally, the disjunct  $\varphi_{bad\_cr}$  checks that there are two adjacent cells in a column which do *not* have the same color on the shared edge; moreover, the rightmost cell is preceded by an incomplete prefix. Formula  $\varphi_{bad\_cr}$  is defined as follows.

$$\begin{aligned} & \left( \theta(1|R_{main}, 2|P_{num}, 3|\Delta_R, 4|R_{cell}, 5|\{\$\}, 6|R_{cell}, 7|P_{num}, 8|\Delta, 9|P_{main}) \vee \right. \\ & \theta(1|R_{main}, 2|P_{num}, 3|\Delta_R, 4|R_{cell}, 5|\{\$\}, 7|P_{num}, 8|\Delta, 9|P_{main}) \vee \\ & \left. \theta(1|R_{main}, 2|P_{num}, 3|\Delta_R, 5|\{\$\}, 6|R_{cell}, 7|P_{num}, 8|\Delta, 9|P_{main}) \right) \wedge \psi_{bad\_cr} \\ & \underbrace{\hspace{10em}}_{\text{mark with } t_2 \text{ and } t_7 \text{ the cell-numbers of two cells } c \text{ and } c' \text{ of two adjacent rows}} \end{aligned}$$

where  $\psi_{bad\_cr}$  asserts that the cells  $c$  and  $c'$  whose cell-numbers are marked by the propositions  $t_2$  and  $t_7$  and whose contents are marked by the propositions  $t_3$  and  $t_8$ , respectively, have the same column number but distinct color on the shared edge.

$$\begin{aligned} \psi_{bad\_cr} = & \bigwedge_{i=1}^{i=n} \bigvee_{b \in \{0,1\}} \underbrace{\left( F((i, b) \wedge t_2) \wedge F((i, b) \wedge t_7) \right)}_{\text{the marked cells } c \text{ and } c' \text{ have the same column number}} \\ & \bigvee_{(d, d') \in \Delta \times \Delta: d_{up} \neq (d')_{down}} \underbrace{\left( F(d \wedge t_3) \wedge F(d' \wedge t_8) \right)}_{\text{the marked cells } c \text{ and } c' \text{ do not have the same color on the shared edge}} \end{aligned}$$

By construction and Proposition III.2,  $\varphi_{bad}$  is a  $\text{THT}_2^1(\mathbf{G})$  formula which can be constructed in polynomial time. Moreover, for all good pseudo-tiling codes  $(\mathbf{H}, \mathbf{T})$  such that  $\mathbf{H} \neq \mathbf{T}$ , if  $(\mathbf{H}, \mathbf{T}) \models \varphi_{bad}$ , then there is no prefix of  $\mathbf{T}$  whose projection over  $P_{main}$  encodes a tiling. On the other hand, by using Remark III.1, for each total good pseudo-tiling code  $(\mathbf{T}, \mathbf{T})$  such that

no prefix of  $T$  encodes a tiling, there exists a good pseudo-tiling code of the form  $(H, T)$  such that  $H \neq T$  and  $(H, T)$  satisfies  $\varphi_{bad}$ . Hence, the first part of Proposition III.3 follows. For the second part, notice that by construction,  $\varphi_{bad}$  is a positive boolean combinations of formulas  $\psi$  such that either  $\psi$  is a  $THT^0$  formula, or  $\psi$  is a formula of Proposition III.2. By the semantics of  $THT$ , for all  $THT^0$  formulas  $\psi$  and interpretations  $(H, T)$ ,  $(H, T) \models \psi$  iff  $H \models_{LTL} \psi$ . Thus, by Proposition III.2, it follows that for all good pseudo-tiling codes  $(H, T)$  for  $THT_2^1(F, G)$  with  $H \neq T$ ,  $(H, T) \models \varphi_{bad}$  iff  $H \models_{LTL} \varphi_{bad}$ . Hence, the result follows.  $\blacksquare$

### B. Proof of Theorem III.1: reductions for the fragments $THT_2^2(G)$ and $THT_2^2(U)$

For the fragments  $THT_2^2(G)$  and  $THT_2^2(U)$ , we give distinct notions of pseudo-tiling code which in turn are different from the one adopted for the fragment  $THT_2^1(F, G)$ . Then, we give corresponding versions of Propositions III.1, III.2 and III.3.

For an LTL interpretation  $T$  over  $P$  and  $i \geq 0$ , we say that  $i$  is an *empty position* of  $T$  if  $T(i) = \emptyset$ .

**Definition VII.1** (Pseudo-tiling codes for  $THT_2^2(G)$  and  $THT_2^2(U)$ ). *Let  $\mathcal{L} \in \{THT_2^2(G), THT_2^2(U)\}$ . An interpretation  $M = (H, T)$  is a pseudo-tiling code for  $\mathcal{L}$  if there is  $L \in \mathbb{N} \cup \{\infty\}$ , with  $L$  being an empty position of  $T$  if  $\mathcal{L} = THT_2^2(U)$ , and  $L$  being  $\infty$  otherwise, such that the following holds:*

- Pseudo-tiling  $T$ -requirement:  $\$ \in T(0)$  and the following holds:
  - for all  $0 \leq i < L$ ,  $T(i) \cap P_{main}$  is a singleton and  $T(i) \cap P_{main} = H(i) \cap P_{main}$ ;
  - there is  $0 \leq i < L$  such that  $d_{final} \in T(i)$ .
- Full  $T$ -requirement: for all  $0 \leq i < L$ ,  $T(i) \cap P_{tag} = P_{tag}$  and  $u \in T(i)$ .
- $H$ -requirement: if  $H \neq T$ , then
  - Case  $\mathcal{L} = THT_2^2(G)$ : there is  $k_\infty \in \mathbb{N} \cup \{\infty\}$  such that (i) for all  $i \leq k_\infty$ ,  $H(i) \cap P_{tag}$  is a singleton and  $u \notin H(i)$ , and (ii) for all  $i > k_\infty$ ,  $H(i) \cap P_{tag} = P_{tag}$  and  $u \in H(i)$ .
  - Case  $\mathcal{L} = THT_2^2(U)$ : for all  $0 \leq i < L$ ,  $H(i) \cap P_{tag} \neq \emptyset$ . Moreover, if there is  $0 \leq i < L$  such that either  $u \in H(i)$  or  $|H(i) \cap P_{tag}| \geq 2$ , then for all  $0 \leq j < L$ ,  $H(j) \cap P_{tag} = P_{tag}$  and  $u \in H(j)$ .

**Definition VII.2** (Slices and good pseudo-tiling codes for  $THT_2^2(G)$  and  $THT_2^2(U)$ ). *Let  $\mathcal{L} \in \{THT_2^2(G), THT_2^2(U)\}$ . For every pseudo-tiling code  $M = (H, T)$  for  $\mathcal{L}$  such that  $H \neq T$ , the slice of  $(H, T)$  is defined as follows:*

- Case  $\mathcal{L} = THT_2^2(G)$ : the slice of  $M$  is  $H$  if  $u \notin H(i)$  for all  $i \geq 0$ ; otherwise, the slice of  $M$  is the maximal prefix of  $H$  whose positions are not labeled by  $u$  (note that such a prefix is non-empty, otherwise  $H = T$ ). Observe that for every position  $i$  of the slice of  $M$ , there is exactly one proposition  $t$  in  $P_{tag}$  such that  $t \in H(i)$ .
- Case  $\mathcal{L} = THT_2^2(U)$ : the slice of  $M$  is the maximal prefix of  $H$  consisting of non-empty positions.

A pseudo-tiling code  $M = (H, T)$  for  $\mathcal{L}$  is good if whenever  $H \neq T$  and  $\mathcal{L} = THT_2^2(U)$ , then for all positions  $i$  of the slice of  $M$ ,  $u \notin H(i)$  and  $H(i) \cap P_{tag}$  is a singleton.

An interpretation  $M = (H, T)$  satisfies the *empty suffix requirement* if there is an empty position  $L$  of  $T$  such that for all  $i > L$  (resp.,  $i < L$ ),  $i$  is an empty position (resp.,  $i$  is not an empty position) of  $T$ . Evidently, by Definitions VII.1 and VII.2, the following holds.



**Remark VII.1.** If  $M$  is a pseudo-tiling code for  $\text{THT}_2^2(U)$  which satisfies the empty suffix requirement, then  $M$  is good.

The notion of pseudo-tiling code for  $\mathcal{L} \in \{\text{THT}_2^2(G), \text{THT}_2^2(U)\}$  can be captured by an  $\mathcal{L}$ -formula.

**Proposition VII.1.** Let  $\mathcal{L} \in \{\text{THT}_2^2(G), \text{THT}_2^2(U)\}$ . Then, one can construct in polynomial time an  $\mathcal{L}$ -formula  $\varphi_{\text{pseudo}}$  such that  $(H, T) \models \varphi_{\text{pseudo}}$  iff  $(H, T)$  is a pseudo-tiling code for  $\mathcal{L}$ .

*Proof:* Case  $\mathcal{L} = \text{THT}_2^2(G)$ : we use the fact that  $(H, T) \models (\neg u \rightarrow u)$  iff  $u \in T(0)$ .

$$\begin{aligned} \varphi_{\text{pseudo}} = & \underbrace{\$ \wedge \mathbf{G} \left( \bigvee_{p \in P_{\text{main}}} (p \wedge \bigwedge_{p' \in P_{\text{main}} \setminus \{p\}} \neg p') \right) \wedge (\neg \mathbf{G} \bigvee_{p \in P_{\text{main}} \setminus \{d_{\text{final}}\}} p) \wedge}_{\text{pseudo-tiling T-requirement}} \\ & \underbrace{(\neg u \rightarrow u) \wedge \mathbf{G} \left( \bigvee_{p \in P_{\text{tag}}} p \right) \wedge \mathbf{G} \left( \left[ \bigvee_{(p, p') \in P_{\text{tag}} : p \neq p'} (p \wedge p') \right] \rightarrow u \right) \wedge \mathbf{G} \left( u \rightarrow \mathbf{G} \left( u \wedge \bigwedge_{p \in P_{\text{tag}}} p \right) \right)}_{\text{Full T-requirement and H-requirement}} \end{aligned}$$

The first three conjuncts in the definition of  $\varphi_{\text{pseudo}}$  evidently capture the pseudo-tiling T-requirement. Moreover, since  $(H, T) \models (\neg u \rightarrow u)$  iff  $u \in T(0)$ , the last four conjuncts ensure the full T-requirement and the H-requirement.

Case  $\mathcal{L} = \text{THT}_2^2(U)$ : let  $\eta_\emptyset = \bigwedge_{p \in P} \neg p$  (characterizing the empty positions).

$$\begin{aligned} \varphi_{\text{pseudo}} = & \underbrace{\$ \wedge \left( \bigvee_{p \in P_{\text{main}}} (p \wedge \bigwedge_{p' \in P_{\text{main}} \setminus \{p\}} \neg p') \cup \eta_\emptyset \right) \wedge \neg \left( \left( \bigvee_{p \in P_{\text{main}} \setminus \{d_{\text{final}}\}} p \right) \cup \eta_\emptyset \right) \wedge}_{\text{pseudo-tiling T-requirement}} \\ & \left( \left( \bigvee_{p \in P_{\text{tag}}} p \right) \cup \eta_\emptyset \right) \wedge \left( \left[ \left( \bigvee_{p \in P} p \right) \cup \bigvee_{(p, p') \in P_{\text{tag}} : p \neq p'} \mathbf{F}(p \wedge p') \right] \rightarrow u \right) \wedge \\ & \underbrace{(\neg u \rightarrow u) \wedge \left[ \left( \left( \bigvee_{p \in P} p \right) \cup u \right) \rightarrow \left( \left( u \wedge \bigwedge_{p \in P_{\text{tag}}} p \right) \cup \eta_\emptyset \right) \right]}_{\text{Full T-requirement and H-requirement}} \end{aligned}$$

■

The following Propositions VII.2 and VII.3 represent the versions of Propositions III.2 and III.3 for the considered fragments  $\text{THT}_2^2(G)$  and  $\text{THT}_2^2(U)$ .

**Proposition VII.2.** Let  $\mathcal{L} \in \{\text{THT}_2^2(G), \text{THT}_2^2(U)\}$ ,  $t_{i_1}, \dots, t_{i_k}$  be distinct propositions in  $P_{\text{tag}}$ , and  $P_1, \dots, P_k$  be non-empty subsets of  $P_{\text{main}}$ . Then, one can construct in polynomial time an  $\mathcal{L}$ -formula  $\theta(i_1|P_1, \dots, i_k|P_k)$  satisfying the following: for all good pseudo-tiling codes  $(H, T)$  for  $\mathcal{L}$  such that  $H \neq T$ ,

$$(H, T) \models \theta(i_1|P_1, \dots, i_k|P_k) \text{ iff}$$

the projection of the slice of  $(H, T)$  over  $P_{\text{tag}}$  is

$$\text{either in } \{t_{i_1}\}^+ \dots \{t_{i_{k-1}}\}^+ \{t_{i_k}\}^\omega \text{ or in } \{t_{i_1}\}^+ \dots \{t_{i_k}\}^+,$$

and for all  $1 \leq j \leq k$ , all the main propositions which label the segment of  $H$  marked by  $t_{i_j}$  are in  $P_j$ .

*Proof: Case  $\mathcal{L} = \text{THT}_2^2(\mathbf{G})$ :* we use the fact that for a pseudo-tiling code  $(H, T)$  for  $\text{THT}_2^2(\mathbf{G})$  such that  $H \neq T$ , a position  $i \geq 0$ , and  $t \in P_{\text{tag}}$ , formula  $t \rightarrow u$  holds at position  $i$  iff either  $t \notin H(i)$  or  $i$  is not a position of the slice of  $(H, T)$ ; moreover, if  $i$  is not a position of the slice of  $(H, T)$ , then  $H(i) \cap P_{\text{tag}} = P_{\text{tag}}$ . Furthermore, for a pseudo-tiling code  $(H, T)$  of  $\text{THT}_2^2(\mathbf{G})$ ,  $H \neq T$  iff  $u \notin H(0)$ .

$$\begin{aligned}
\theta(i_1|P_1, \dots, i_k|P_k) = & \underbrace{\bigwedge_{t \in P_{\text{tag}} \setminus \{t_{i_1}, \dots, t_{i_k}\}} \mathbf{G}(t \rightarrow u)}_{\text{the slice is only marked by tag propositions in } \{t_{i_1}, \dots, t_{i_k}\}} \quad \wedge \\
& \underbrace{\bigwedge_{j=1}^{j=k} \left( [\mathbf{G}(t_{i_j} \rightarrow u)] \longrightarrow u \right)}_{\text{every tag } t_{i_j} \text{ (} j = 1, \dots, k \text{) marks some position of the slice}} \quad \wedge \\
& \underbrace{\bigwedge_{j=1}^{j=k} \mathbf{G} \left( (t_{i_j} \rightarrow u) \vee \bigvee_{p \in P_j} p \right)}_{\text{the positions of the slice marked by } t_{i_j} \text{ are labeled by main propositions in } P_j} \quad \wedge \\
& \underbrace{\bigwedge_{j=1}^{j=k} \mathbf{G} \left( t_{i_j} \rightarrow \mathbf{G} \left( \bigvee_{r=j}^{r=k} t_{i_r} \right) \right)}_{\text{the tags } t_{i_j} \text{ mark the slice according to the order } t_{i_1}, \dots, t_{i_k}}
\end{aligned}$$

*Case  $\mathcal{L} = \text{THT}_2^2(\mathbf{U})$ :* we use the fact that for a good pseudo-tiling code  $(H, T)$  for  $\text{THT}_2^2(\mathbf{U})$  such that  $H \neq T$ ,  $u \in T(0)$ ,  $u \notin H(0)$  and for all the positions  $i$  of the slice of  $(H, T)$ ,  $H(i) \cap P_{\text{tag}}$  is a singleton. In order to define the  $\text{THT}_2^2(\mathbf{U})$ -formula  $\theta(i_1|P_1, \dots, i_k|P_k)$ , we use for all  $t, t' \in P_{\text{tag}}$  and for all implication-free propositional formulas  $\xi$ , the following auxiliary  $\text{THT}_2^2(\mathbf{U})$ -formulas  $\psi(t, t')$  and  $\phi(\xi)$

$$\begin{aligned}
\psi(t, t') &= \left( \bigvee_{p \in P} p \right) \mathbf{U} \left( t \wedge \left[ \left( \bigvee_{p \in P} p \right) \mathbf{U} t' \right] \right) \\
\phi(\xi) &= \left( \bigvee_{p \in P} p \right) \mathbf{U} \xi
\end{aligned}$$

Formula  $\psi(t, t')$  asserts that along the slice of the given good pseudo-tiling code for  $\text{THT}_2^2(\mathbf{U})$ , there is a position marked by  $t$  followed by a position marked by  $t'$ . Formula  $\phi(\xi)$  requires that there is a position along the slice, where  $\xi$  holds. The  $\text{THT}_2^2(\mathbf{U})$ -formula  $\theta(i_1|P_1, \dots, i_k|P_k)$  is defined as follows.

$$\theta(i_1|P_1, \dots, i_k|P_k) = \underbrace{\left( \bigwedge_{j=1}^{j=k-1} \psi(t_{i_j}, t_{i_{j+1}}) \right)}_{\text{partial order requirement}} \wedge \left( \psi(i_1|P_1, \dots, i_k|P_k) \longrightarrow u \right)$$

$$\begin{aligned}
\psi(i_1|P_1, \dots, i_k|P_k) = & \underbrace{\left( \bigvee_{t \in P_{tag} \setminus \{t_{i_1}, \dots, t_{i_k}\}} \phi(t) \right)}_{\text{some position in the slice is marked by some } t \in P_{tag} \setminus \{t_{i_1}, \dots, t_{i_k}\}} \vee \\
& \underbrace{\left( \bigvee_{(t, t') \in P_{tag} \times P_{tag}: t \neq t'} (\psi(t, t') \wedge \psi(t', t)) \right)}_{\text{in the slice, a } t'\text{-marked position occurs between two } t\text{-marked positions with } t \neq t'} \vee \\
& \underbrace{\left( \bigvee_{j=1}^{j=k} \bigvee_{p \in P_{main} \setminus P_j} \phi(t_{i_j} \wedge p) \right)}_{\text{for some } 1 \leq j \leq k, \text{ a } t_{i_j}\text{-marked position in the slice is labeled by a } (P_{main} \setminus P_j)\text{-proposition}}
\end{aligned}$$

■

**Proposition VII.3.** *Let  $\mathcal{L} \in \{THT_2^2(\mathbf{G}), THT_2^2(\mathbf{U})\}$ . Then, one can construct in polynomial time an  $\mathcal{L}$ -formula  $\varphi_{bad}$  such that for all total interpretations  $(T, T)$  which are pseudo-tiling codes for  $\mathcal{L}$ , there exists a good pseudo-tiling code for  $\mathcal{L}$  of the form  $(H, T)$  with  $H \neq T$  and satisfying  $\varphi_{bad}$  iff there is no prefix of  $T$  whose projection over  $P_{main}$  encodes a tiling.*

*Proof:* The  $\mathcal{L}$ -formula  $\varphi_{bad}$  is defined as follows

$$\begin{aligned}
\varphi_{bad} = & \varphi_{bad\_in} \vee \varphi_{bad\_ord} \vee \varphi_{bad\_acc} \vee \varphi_{bad\_cell} \vee \varphi_{bad\_first} \vee \varphi_{bad\_last} \vee \\
& \varphi_{bad\_inc} \vee \varphi_{bad\_rr} \vee \varphi_{bad\_cr}
\end{aligned}$$

where for a total pseudo-tiling code  $(T, T)$  for  $\mathcal{L}$ , the different disjuncts in the definition of  $\varphi_{bad}$  have the same intended meaning as the homonym disjuncts in the proof of Proposition III.3. In particular, they capture all the possible conditions such that no prefix of  $T$  encodes a tiling iff some of these conditions is satisfied. The construction of such disjuncts exploits the formulas  $\theta(i_1|P_1, \dots, i_k|P_k)$  of Proposition VII.2. In particular, all the above disjuncts – except  $\varphi_{bad\_inc}$  and  $\varphi_{bad\_cr}$  – are defined as the homonym disjuncts in the proof of Proposition III.3, but we use the formulas  $\theta(i_1|P_1, \dots, i_k|P_k)$  of Proposition VII.2 instead of the formulas of Proposition III.2. The construction of  $\varphi_{bad\_inc}$  and  $\varphi_{bad\_cr}$  is as follows. Recall from the proof of Proposition III.3 that for an LTL interpretation  $T$  over  $P$ , a prefix of  $T$  is *incomplete* if it has no position labeled by  $d_{final}$ .

We use the following short-hands:

$$\begin{aligned}
R_{main} &= P_{main} \setminus \{d_{final}\} \\
P_{cell} &= P_{main} \setminus \{\$ \} \\
R_{cell} &= P_{cell} \setminus \{d_{final}\} \\
\Delta_R &= \Delta \setminus \{d_{final}\}
\end{aligned}$$

For a good pseudo-tiling code  $(H, T)$  for  $\mathcal{L}$  such that  $H \neq T$ , the disjunct  $\varphi_{bad\_inc}$  selects along the slice adjacent cells in a row whose column numbers are not consecutive; moreover, the rightmost cell is preceded by an incomplete prefix. In order to define  $\varphi_{bad\_inc}$ , we use the following auxiliary  $\mathcal{L}$ -formulas  $\phi(p, t)$  where  $p \in P_{main}$  and  $t \in P_{tag}$ :

- Case  $\mathcal{L} = \text{THT}_2^2(\mathbf{G})$ :

$$\phi(p, t) = \left( \mathbf{G}[(t \rightarrow u) \vee \bigvee_{p' \in P_{\text{main}} \setminus \{p\}} p'] \right) \rightarrow u$$

- Case  $\mathcal{L} = \text{THT}_2^2(\mathbf{U})$ :  $\phi(p, t) = (\bigvee_{p' \in P} p') \vee (p \wedge t)$

It is easy to check that for a good pseudo-tiling code  $(\mathbf{H}, \mathbf{T})$  for  $\mathcal{L}$  such that  $\mathbf{H} \neq \mathbf{T}$ ,  $(\mathbf{H}, \mathbf{T}) \models \phi(p, t)$  iff there is a position  $i$  of the slice of  $(\mathbf{H}, \mathbf{T})$  marked by  $t$  and where  $p$  holds. The formula  $\varphi_{\text{bad\_inc}}$  is defined as follows:

$$\begin{aligned} \varphi_{\text{bad\_inc}} = & \left( \theta(1|R_{\text{main}}, 2|\Delta_R \cup \{\$\}, 3|P_{\text{num}}, 4|\Delta_R, 5|P_{\text{num}}, 6|\Delta, 7|P_{\text{main}}) \vee \right. \\ & \left. \underbrace{\theta(1|\{\$\}, 3|P_{\text{num}}, 4|\Delta_R, 5|P_{\text{num}}, 6|\Delta, 7|P_{\text{main}})}_{\text{mark by } t_3 \text{ and } t_5 \text{ the cell-numbers of two adjacent cells in a row}} \right) \wedge \\ & \psi_{\text{bad\_inc}} \end{aligned}$$

where  $\psi_{\text{bad\_inc}}$  uses the above formulas  $\phi(p, t)$  and asserts that the cell numbers marked by  $t_3$  and  $t_5$ , respectively, are not consecutive.

$$\begin{aligned} \psi_{\text{bad\_inc}} = & \bigwedge_{i=1}^{i=n} \left( \phi((i, 1), t_3) \wedge \phi((i, 0), t_5) \right) \vee \bigvee_{i=1}^{i=n} \left( \phi((i, 0), t_3) \wedge \phi((i, 1), t_5) \wedge \right. \\ & \left. \left[ \bigvee_{j=1}^{j=i-1} (\phi((j, 0), t_3) \wedge \phi((j, 1), t_5)) \vee \bigvee_{j=i+1}^{j=n} \bigvee_{b, b' \in \{0, 1\}: b \neq b'} (\phi((j, b), t_3) \wedge \phi((j, b'), t_5)) \right] \right) \end{aligned}$$

Finally, the disjunct  $\varphi_{\text{bad\_cr}}$  checks that, along the slice, there are two adjacent cells in a column which do *not* have the same color on the shared edge; moreover, the rightmost cell is preceded by an incomplete prefix. Formula  $\varphi_{\text{bad\_cr}}$  is defined as follows.

$$\begin{aligned} & \left( \theta(1|R_{\text{main}}, 2|P_{\text{num}}, 3|\Delta_R, 4|R_{\text{cell}}, 5|\{\$\}, 6|R_{\text{cell}}, 7|P_{\text{num}}, 8|\Delta, 9|P_{\text{main}}) \vee \right. \\ & \theta(1|R_{\text{main}}, 2|P_{\text{num}}, 3|\Delta_R, 4|R_{\text{cell}}, 5|\{\$\}, 7|P_{\text{num}}, 8|\Delta, 9|P_{\text{main}}) \vee \\ & \left. \theta(1|R_{\text{main}}, 2|P_{\text{num}}, 3|\Delta_R, 5|\{\$\}, 6|R_{\text{cell}}, 7|P_{\text{num}}, 8|\Delta, 9|P_{\text{main}}) \right) \wedge \psi_{\text{bad\_cr}} \\ & \underbrace{\hspace{10em}}_{\text{mark with } t_2 \text{ and } t_7 \text{ the cell-numbers of two cells } c \text{ and } c' \text{ of two adjacent rows}} \end{aligned}$$

where  $\psi_{\text{bad\_cr}}$  asserts that the cells  $c$  and  $c'$  whose cell-numbers are marked by the propositions  $t_2$  and  $t_7$  and whose contents are marked by the propositions  $t_3$  and  $t_8$ , respectively, have the same column number but distinct color on the shared edge. For the construction of  $\psi_{\text{bad\_cr}}$ , we use the  $\mathcal{L}$ -formulas  $\phi(p, t)$  (where  $p \in P_{\text{main}}$  and  $t \in P_{\text{tag}}$ ) exploited in the construction of  $\varphi_{\text{bad\_inc}}$ .

$$\begin{aligned} \psi_{\text{bad\_cr}} = & \bigwedge_{i=1}^{i=n} \bigvee_{b \in \{0, 1\}} \left( \phi((i, b), t_2) \wedge \phi((i, b), t_7) \right) \wedge \\ & \underbrace{\hspace{10em}}_{\text{the marked cells } c \text{ and } c' \text{ have the same column number}} \\ & \bigvee_{\substack{(d, d') \in \Delta \times \Delta: d_{\text{up}} \neq (d')_{\text{down}}}} \left( \phi(d, t_3) \wedge \phi(d', t_8) \right) \\ & \underbrace{\hspace{10em}}_{\text{the marked cells } c \text{ and } c' \text{ do not have the same color on the shared edge}} \end{aligned}$$

■

By using Propositions VII.1 and VII.3, we prove the following result from which Theorem III.1 for the fragments  $\text{THT}_2^2(\mathbf{G})$  and  $\text{THT}_2^2(\mathbf{U})$  directly follows.

**Lemma VII.1.** *Let  $\mathcal{L} \in \{\text{THT}_2^2(\mathbf{G}), \text{THT}_2^2(\mathbf{U})\}$ . Then, one can construct in polynomial time an  $\mathcal{L}$ -formula  $\varphi_{\mathcal{I}}$  such that there is a temporal equilibrium model of  $\varphi_{\mathcal{I}}$  iff there is a tiling of  $\mathcal{I}$ .*

*Proof:* Let  $\varphi_{\text{pseudo}}$  be the  $\mathcal{L}$ -formula of Proposition VII.1 and  $\varphi_{\text{bad}}$  be the  $\mathcal{L}$ -formula of Proposition VII.3. Then:

$$\varphi_{\mathcal{I}} = \varphi_{\text{pseudo}} \wedge (u \vee \varphi_{\text{bad}})$$

Now, we prove that the construction is correct. First, assume that there exists a temporal equilibrium model  $(\mathbf{T}, \mathbf{T})$  of  $\varphi_{\mathcal{I}}$ . By construction of  $\varphi_{\mathcal{I}}$  and Proposition VII.1,  $(\mathbf{T}, \mathbf{T})$  is a pseudo-tiling code for  $\mathcal{L}$ . If no prefix of  $\mathbf{T}$  encodes a tiling, by Proposition VII.3, there exists  $\mathbf{H} \sqsubset \mathbf{T}$  such that  $(\mathbf{H}, \mathbf{T}) \models \varphi_{\text{bad}}$  and  $(\mathbf{H}, \mathbf{T})$  is a good pseudo-tiling code for  $\mathcal{L}$ ; hence, by Proposition VII.1,  $(\mathbf{H}, \mathbf{T})$  satisfies  $\varphi_{\mathcal{I}}$ , which contradicts the assumption that  $(\mathbf{T}, \mathbf{T})$  is a temporal equilibrium model. Thus, some prefix of  $\mathbf{T}$  encodes a tiling, and the result follows.

Now, assume that there exists a tiling  $f$  of  $\mathcal{I}$ . Assume that  $\mathcal{L} = \text{THT}_2^2(\mathbf{U})$  (the other case being simpler). Let  $(\mathbf{T}, \mathbf{T})$  be any pseudo-tiling code for  $\text{THT}_2^2(\mathbf{U})$  satisfying the *empty suffix requirement* such that the projection of some prefix of  $\mathbf{T}$  over  $P_{\text{main}}$  is an encoding of  $f$ . Note that such a  $(\mathbf{T}, \mathbf{T})$  exists. Since  $u \in \mathbf{T}(0)$ , by construction and Proposition VII.1,  $(\mathbf{T}, \mathbf{T})$  satisfies  $\varphi_{\mathcal{I}}$ . We assume that  $(\mathbf{T}, \mathbf{T})$  is not an equilibrium model and derive a contradiction, hence, the result follows. Thus, there is  $\mathbf{H} \sqsubset \mathbf{T}$  such that  $(\mathbf{H}, \mathbf{T}) \models \varphi_{\mathcal{I}}$ . By construction and Proposition VII.1,  $(\mathbf{H}, \mathbf{T})$  is a pseudo-tiling code for  $\text{THT}_2^2(\mathbf{U})$ . Since  $(\mathbf{T}, \mathbf{T})$  satisfies the *empty suffix requirement*,  $(\mathbf{H}, \mathbf{T})$  satisfies the empty suffix requirement as well. Thus, by Remark VII.1,  $(\mathbf{H}, \mathbf{T})$  is a good pseudo-tiling code, and in particular,  $u \notin \mathbf{H}(0)$ . Since  $(\mathbf{H}, \mathbf{T}) \models \varphi_{\mathcal{I}}$ , by construction,  $(\mathbf{H}, \mathbf{T}) \models \varphi_{\text{bad}}$ . Thus, by Proposition VII.3, there is no prefix of  $\mathbf{T}$  which encodes a tiling. This contradicts the hypothesis, and we are done.

■

### C. Proof of Proposition III.4

**Proposition III.4.** *One can construct in polynomial time a  $\text{THT}_1^2(F, G)$  formula  $\varphi_{\text{pseudo}}$  such that  $(H, T) \models \varphi_{\text{pseudo}}$  iff  $(H, T)$  is a pseudo-tiling code for  $\text{THT}_1^2(F, G)$ .*

*Proof:* We use the fact that  $(H, T) \models (\neg u \rightarrow u)$  iff  $u \in T(0)$ . The  $\text{THT}_1^2(F, G)$  formula  $\varphi_{\text{pseudo}}$  is defined as follows:

$$\varphi_{\text{pseudo}} = (\neg u \rightarrow u) \wedge \varphi_T \wedge \varphi_{\text{full}} \wedge \varphi_H$$

where  $\varphi_T$ ,  $\varphi_{\text{full}}$ , and  $\varphi_H$  are  $\text{THT}_1^1(F, G)$  formulas, and:  $\varphi_T$  ensures the pseudo-tiling T-requirement,  $\varphi_{\text{full}}$  together with the conjunct  $\neg u \rightarrow u$  ensures the full T-requirement, and  $\varphi_H$  together with the conjuncts  $\neg u \rightarrow u$  and  $\varphi_{\text{full}}$  guarantees the H-requirement.

$$\begin{aligned} \varphi_T = & \mathbf{G} \bigvee_{d \in \Delta} \left( d \wedge \bigwedge_{d' \in \Delta \setminus \{d\}} \neg d' \right) \wedge \bigwedge_{i=1}^{i=n} \bigwedge_{\tau \in \{r, c\}} \mathbf{G} \bigvee_{b \in \{0, 1\}} \left( (\tau, i, b) \wedge \neg(\tau, i, 1 - b) \right) \wedge \\ & \underbrace{\mathbf{F} \left( d_{\text{init}} \wedge \bigwedge_{i=1}^{i=n} \bigwedge_{\tau \in \{r, c\}} (\tau, i, 0) \right)}_{\text{initialization}} \wedge \underbrace{\mathbf{F} \left( d_{\text{final}} \wedge \bigwedge_{i=1}^{i=n} \bigwedge_{\tau \in \{r, c\}} (\tau, i, 1) \right)}_{\text{acceptance}} \\ \varphi_{\text{full}} = & \mathbf{F} u \rightarrow \mathbf{G} \left( u \wedge \bigwedge_{p \in P_{\text{tag}}} p \right) \end{aligned}$$

$$\varphi_H = \mathbf{G} \left( t_1 \vee t_2 \vee t_3 \vee \bigwedge_{i=1}^{i=n} \bigwedge_{\tau \in \{r, c\}} \bigvee_{b \in \{0, 1\}} \overline{(\tau, i, b)} \right) \wedge \left( \varphi_{\text{bad}_H} \rightarrow u \right)$$

$$\begin{aligned} \varphi_{\text{bad}_H} = & \left[ \mathbf{F}(t_1 \vee t_2 \vee t_3) \wedge \mathbf{F} \left( \bigvee_{t \in P_{\text{tag}} \setminus \{t_1, t_2, t_3\}} t \right) \right] \vee \mathbf{F} \left[ \bigvee_{t, t' \in \{t_1, t_2, t_3\}: t \neq t'} (t \wedge t') \right] \vee \\ & \bigvee_{i=1}^{i=n} \bigvee_{\tau \in \{r, c\}} [(\mathbf{F} \overline{(\tau, i, 0)}) \wedge (\mathbf{F} \overline{(\tau, i, 1)})] \end{aligned}$$

■

### D. Proof of Theorem III.2 for the fragments $\text{THT}_1^2(\mathbf{U})$ and $\text{THT}_1^2(\mathbf{R})$

*Encoding of tilings:* The notions of cell-codes and cell-number codes (over  $P_{\text{tag}}$ ) are defined as for the reduction given for the fragment  $\text{THT}_1^2(F, G)$ . However, a tiling  $f : [0, 2^n - 1] \times [0, 2^n - 1] \rightarrow \Delta$  (of the given instance  $\mathcal{I}$ ) is encoded by finite words (and not infinite words)  $w$  over  $2^{P_{\text{main}}}$  satisfying the following, where  $|w|$  denotes the length of  $w$ :

- for all  $i, j \in [0, 2^n - 1]$ , there is  $0 \leq h < |w|$  such that  $w(h)$  is the cell-code of the  $(i, j)^{\text{th}}$  cell of  $f$ ;
- for all  $0 \leq h < |w|$ ,  $w(h)$  encodes the  $(i, j)^{\text{th}}$  cell of  $f$  for some  $i, j \in [0, 2^n - 1]$ .

*Reductions for  $THT_1^2(U)$  and  $THT_1^2(R)$ :* for these two fragments, we give two slightly different notions of pseudo-tiling code which in turn are different from the one adopted for the fragment  $THT_1^2(F, G)$ . Then, we provide corresponding versions of Propositions III.4 and III.5.

Recall from Appendix VII-B that for an LTL interpretation  $T$  over  $P$  and for  $i \geq 0$ ,  $i$  is an *empty position* of  $T$  if  $T(i) = \emptyset$ . An interpretation  $M = (H, T)$  satisfies the *empty suffix requirement* if there is an empty position  $L$  of  $T$  such that for all  $i > L$  (resp.,  $i < L$ ),  $i$  is an empty position (resp.,  $i$  is not an empty position) of  $T$ .

**Definition VII.3** (Pseudo-tiling codes for  $THT_1^2(U)$ ). *An interpretation  $M = (H, T)$  is a pseudo-tiling code for  $THT_1^2(U)$  if there is an empty position  $L$  of  $T$  such that the following holds:*

- Pseudo-tiling  $T$ -requirement: *for all  $0 \leq i < L$ ,  $T(i) \cap P_{main}$  is a cell-code and  $H(i) \cap P_{main} = T(i) \cap P_{main}$ . Moreover,*
  - *there is  $0 \leq i < L$  such that  $T(i) \cap P_{main}$  has row-number 0, column-number 0 and  $d_{init} \in T(i)$  (initialization);*
  - *there is  $0 \leq i < L$  such that  $T(i) \cap P_{main}$  has row-number  $2^n - 1$ , column-number  $2^n - 1$ , and  $d_{final} \in T(i)$  (acceptance).*
- Full  $T$ -requirement: *for all  $0 \leq i < L$ ,  $T(i) \cap P_{tag} = P_{tag}$  and  $u \in T(i)$ .*
- $H$ -requirement: *for all  $0 \leq i < L$ , either  $H(i) \cap \{t_1, t_2, t_3\} \neq \emptyset$ , or there is a cell-number code  $P' \subseteq P_{tag}$  such that  $P' \subseteq H(i)$ . Moreover, if the following goodness condition is not satisfied, then for all  $0 \leq i < L$ ,  $H(i) = T(i)$ :*  
*Goodness condition: for all  $1 \leq i < L$ ,  $u \notin H(i)$  and*
  - *either there is a cell-number code  $P' \subseteq P_{tag}$  such that for all  $0 \leq i < L$ ,  $H(i) \cap P_{tag} = P'$ ;*
  - *or for all  $0 \leq i < L$ ,  $H(i) \cap P_{tag}$  is a singleton contained in  $\{t_1, t_2, t_3\}$ .*

*The slice of  $(H, T)$  is the prefix of  $H$  of length  $L$  (i.e., the maximal prefix of  $H$  consisting of non-empty positions of  $T$ ).*

**Definition VII.4** (Pseudo-tiling codes for  $THT_1^2(R)$ ). *An interpretation  $M = (H, T)$  is a pseudo-tiling code for  $THT_1^2(R)$  if there is an empty position  $L$  of  $T$  such that the following holds:*

- Pseudo-tiling  $T$ -requirement: *for all  $i \geq 0$ , either  $i$  is an empty position of  $T$ , or  $T(i) \cap P_{main}$  is a cell-code and  $H(i) \cap P_{main} = T(i) \cap P_{main}$ . Moreover,*
  - *there is  $0 \leq i < L$  such that  $T(i) \cap P_{main}$  has row-number 0, column-number 0 and  $d_{init} \in T(i)$  (initialization);*
  - *there is  $0 \leq i < L$  such that  $T(i) \cap P_{main}$  has row-number  $2^n - 1$ , column-number  $2^n - 1$ , and  $d_{final} \in T(i)$  (acceptance).*
- Full  $T$ -requirement: *for all non-empty positions  $i$  of  $T$ ,  $T(i) \cap P_{tag} = P_{tag}$  and  $u \in T(i)$ .*
- $H$ -requirement: *for all non-empty positions  $i$  of  $T$ , either  $H(i) \cap \{t_1, t_2, t_3\} \neq \emptyset$ , or there is a cell-number code  $P' \subseteq P_{tag}$  such that  $P' \subseteq H(i)$ . Moreover, if the goodness condition is not satisfied, then  $H = T$ , where the goodness condition is defined as in Definition VII.3.*

*The slice of  $(H, T)$  is defined as in Definition VII.3.*

A pseudo-tiling code for  $THT_1^2(U)$  (resp.,  $THT_1^2(R)$ )  $M = (H, T)$  is *good* if whenever  $H \neq T$ , then  $M$  satisfies the goodness condition. Evidently, the following holds.

**Remark VII.2.** *If  $M$  is a pseudo-tiling code for  $THT_1^2(R)$ , then  $M$  is good. Moreover, if  $M$  is a pseudo-tiling code for  $THT_1^2(U)$  which satisfies the empty suffix requirement, then  $M$  is good.*

The following Propositions VII.4 and VII.5 represent the variants for the fragments  $\text{THT}_1^2(\mathbf{R})$  and  $\text{THT}_1^2(\mathbf{U})$  of Propositions III.4 and III.5.

**Proposition VII.4.** *Let  $\mathcal{L} \in \{\text{THT}_1^2(\mathbf{U}), \text{THT}_1^2(\mathbf{R})\}$ . Then, one can construct in polynomial time an  $\mathcal{L}$ -formula  $\varphi_{\text{pseudo}}$  such that  $(H, T) \models \varphi_{\text{pseudo}}$  iff  $(H, T)$  is a pseudo-tiling code for  $\mathcal{L}$ .*

*Proof:* The  $\mathcal{L}$ -formula  $\varphi_{\text{pseudo}}$  is defined as follows:

$$\varphi_{\text{pseudo}} = (\neg u \rightarrow u) \wedge \varphi_{\text{T}} \wedge \varphi_{\text{full}} \wedge \varphi_{\text{H}}$$

where  $\varphi_{\text{T}}$ ,  $\varphi_{\text{full}}$ , and  $\varphi_{\text{H}}$  are  $\mathcal{L}$  formulas, and:  $\varphi_{\text{T}}$  ensures the pseudo-tiling T-requirement for  $\mathcal{L}$ ,  $\varphi_{\text{full}}$  together with the conjunct  $\neg u \rightarrow u$  ensures the full T-requirement for  $\mathcal{L}$ , and  $\varphi_{\text{H}}$  together with the conjuncts  $\neg u \rightarrow u$  and  $\varphi_{\text{full}}$  guarantees the H-requirement for  $\mathcal{L}$ .

We use the propositional formula  $\eta_0 = \bigwedge_{p \in P} \neg p$  (which characterizes the empty positions).

**Case  $\mathcal{L} = \text{THT}_1^2(\mathbf{U})$ :** for each propositional formula  $\xi$ , let  $\psi(\xi)$  be the  $\text{THT}_1^2(\mathbf{U})$ -formula given by

$$\psi(\xi) = (\bigvee_{p \in P} p) \cup \xi$$

Then:

$$\begin{aligned} \varphi_{\text{T}} &= \left[ \left( \bigvee_{d \in \Delta} (d \wedge \bigwedge_{d' \in \Delta \setminus \{d\}} \neg d') \right) \cup \eta_0 \right] \wedge \\ &\quad \left[ \left( \bigwedge_{i=1}^{i=n} \bigwedge_{\tau \in \{r, c\}} \bigvee_{b \in \{0, 1\}} ((\tau, i, b) \wedge \neg(\tau, i, 1 - b)) \right) \cup \eta_0 \right] \wedge \\ &\quad \underbrace{\psi\left(d_{\text{init}} \wedge \bigwedge_{i=1}^{i=n} \bigwedge_{\tau \in \{r, c\}} (\tau, i, 0)\right)}_{\text{initialization}} \wedge \underbrace{\psi\left(d_{\text{final}} \wedge \bigwedge_{i=1}^{i=n} \bigwedge_{\tau \in \{r, c\}} (\tau, i, 1)\right)}_{\text{acceptance}} \\ \varphi_{\text{full}} &= \psi(u) \rightarrow \left( (u \wedge \bigwedge_{p \in P_{\text{tag}}} p) \cup \eta_0 \right) \\ \varphi_{\text{H}} &= \left[ \left( t_1 \vee t_2 \vee t_3 \vee \bigwedge_{i=1}^{i=n} \bigwedge_{\tau \in \{r, c\}} \bigvee_{b \in \{0, 1\}} \overline{(\tau, i, b)} \right) \cup \eta_0 \right] \wedge (\varphi_{\text{bad\_H}} \rightarrow u) \\ \varphi_{\text{bad\_H}} &= \left[ \psi(t_1 \vee t_2 \vee t_3) \wedge \psi\left( \bigvee_{t \in P_{\text{tag}} \setminus \{t_1, t_2, t_3\}} t \right) \right] \vee \psi\left[ \bigvee_{t, t' \in \{t_1, t_2, t_3\}: t \neq t'} (t \wedge t') \right] \vee \\ &\quad \bigvee_{i=1}^{i=n} \bigvee_{\tau \in \{r, c\}} [\psi(\overline{(\tau, i, 0)}) \wedge \psi(\overline{(\tau, i, 1)})] \end{aligned}$$

**Case  $\mathcal{L} = \text{THT}_1^2(\mathbf{R})$ :** for each propositional formula  $\xi$ , let  $\psi(\xi)$  be the  $\text{THT}_1^2(\mathbf{R})$ -formula given by

$$\psi(\xi) = \xi \text{ R } \bigvee_{p \in P} p$$



Then:

$$\begin{aligned}
\varphi_T &= (\neg \mathbf{G} \bigvee_{p \in P} p) \wedge \mathbf{G} \left( \eta_0 \vee \bigvee_{d \in \Delta} (d \wedge \bigwedge_{d' \in \Delta \setminus \{d\}} \neg d') \right) \wedge \\
&\quad \mathbf{G} \left( \eta_0 \vee \bigwedge_{i=1}^{i=n} \bigwedge_{\tau \in \{r, c\}} \bigvee_{b \in \{0, 1\}} ((\tau, i, b) \wedge \neg(\tau, i, 1 - b)) \right) \wedge \\
&\quad \underbrace{\psi \left( d_{init} \wedge \bigwedge_{i=1}^{i=n} \bigwedge_{\tau \in \{r, c\}} (\tau, i, 0) \right)}_{\text{initialization}} \wedge \underbrace{\psi \left( d_{final} \wedge \bigwedge_{i=1}^{i=n} \bigwedge_{\tau \in \{r, c\}} (\tau, i, 1) \right)}_{\text{acceptance}} \\
\varphi_{full} &= \psi(u) \rightarrow \mathbf{G} \left( \eta_0 \vee (u \wedge \bigwedge_{p \in P_{tag}} p) \right) \\
\varphi_H &= \mathbf{G} \left( \eta_0 \vee t_1 \vee t_2 \vee t_3 \vee \bigwedge_{i=1}^{i=n} \bigwedge_{\tau \in \{r, c\}} \bigvee_{b \in \{0, 1\}} \overline{(\tau, i, b)} \right) \wedge (\varphi_{bad\_H} \rightarrow u) \\
\varphi_{bad\_H} &= \left[ \psi(t_1 \vee t_2 \vee t_3) \wedge \psi \left( \bigvee_{t \in P_{tag} \setminus \{t_1, t_2, t_3\}} t \right) \right] \vee \psi \left[ \bigvee_{t, t' \in \{t_1, t_2, t_3\}: t \neq t'} (t \wedge t') \right] \vee \\
&\quad \bigvee_{i=1}^{i=n} \bigvee_{\tau \in \{r, c\}} [\psi(\overline{(\tau, i, 0)}) \wedge \psi(\overline{(\tau, i, 1)})]
\end{aligned}$$

■

**Proposition VII.5.** *Let  $\mathcal{L} \in \{THT_1^2(U), THT_1^2(R)\}$ . Then, one can construct in polynomial time an  $\mathcal{L}$ -formula  $\varphi_{bad}$  such that for all total interpretations  $M = (T, T)$  which are pseudo-tiling codes for  $\mathcal{L}$ , there exists a good pseudo-tiling code for  $\mathcal{L}$  of the form  $(H, T)$  with  $H \neq T$  and satisfying  $\varphi_{bad}$  iff the projection of the slice of  $M$  over  $P_{main}$  does not encode a tiling.*

*Proof:* First, we define some auxiliary formulas. As in the proof of Proposition VII.4, for each implication-free propositional formula  $\xi$ , we consider the following  $\mathcal{L}$ -formula  $\psi(\xi)$ .

- Case  $\mathcal{L} = THT_1^2(U)$ :  $\psi(\xi) = (\bigvee_{p \in P} p) \cup \xi$
- Case  $\mathcal{L} = THT_1^2(R)$ :  $\psi(\xi) = \xi \mathbf{R} (\bigvee_{p \in P} p)$

For a pseudo-tiling code  $M$  for  $\mathcal{L}$ , the  $\mathcal{L}$ -formula  $\psi(\xi)$  asserts that there is a position of the slice of  $M$ , where  $\xi$  holds.

Moreover, for all  $t, t' \in \{t_1, t_2, t_3\}$  and  $\tau \in \{r, c\}$ , we construct an  $\mathcal{L}$ -formula  $\phi(t, t', \tau)$  such that for each good pseudo-tiling code  $(H, T)$  for  $\mathcal{L}$  with  $H \neq T$ ,  $(H, T) \models \phi(t, t', r)$  (resp.,  $(H, T) \models \phi(t, t', c)$ ) iff for all the positions of the slice of  $(H, T)$  which are marked by the propositions  $t$  and  $t'$ , the associated cell-codes have the same row-number (resp., column-number).

$$\phi(t, t', \tau) = \left( \bigvee_{i=1}^{i=n} \left[ \psi((t \vee t') \wedge (\tau, i, 0)) \wedge \psi((t \vee t') \wedge (\tau, i, 1)) \right] \right) \rightarrow u$$

Then, the  $\mathcal{L}$ -formula  $\varphi_{bad}$  consists of four disjuncts which are defined similarly to the disjuncts in the proof of Proposition III.5 but for their construction, we use the above formulas  $\psi(\xi)$  and  $\phi(t, t', \tau)$ . ■

Fix  $\mathcal{L} \in \{\text{THT}_1(\text{U}), \text{THT}_1(\text{R})\}$ . Let  $\varphi_{\mathcal{I}}$  be the  $\mathcal{L}$ -formula defined as follows:

$$\varphi_{\mathcal{I}} = \varphi_{pseudo} \wedge (u \vee \varphi_{bad})$$

where  $\varphi_{pseudo}$  is the  $\mathcal{L}$ -formula of Proposition VII.4 and  $\varphi_{bad}$  is the  $\mathcal{L}$ -formula of Proposition VII.5. By Propositions VII.4 and VII.5,  $\varphi_{\mathcal{I}}$  can be constructed in polynomial time. Moreover, by Propositions VII.4 and VII.5, we easily deduce the following result, hence, Theorem III.2 for the fragment  $\mathcal{L} \in \{\text{THT}_1^2(\text{U}), \text{THT}_1^2(\text{R})\}$  directly follows.

**Lemma VII.2** (Correctness of the construction). *There exists a temporal equilibrium model of  $\varphi_{\mathcal{I}}$  iff there exists a tiling of  $\mathcal{I}$ .*

*Proof:* The proof is similar to the one of Lemma VII.1 in Appendix VII-B, and we omit the details here. ■

### E. Proof of Lemma III.3

In order to prove Lemma III.3, we exploit a notion of similarity and contraction for interpretations.

**Definition VII.5** (Similarity and contraction). Let  $M$  and  $M'$  be two interpretations. We say that  $M'$  is a *simulation* of  $M$  if:

- $M(0) = M'(0)$  and  $M(1) = M'(1)$ ;
- for all  $i \geq 0$ , there is  $i' \geq 0$  such that  $M'(i') = M(i)$  and for all  $k' \in [0, i' - 1]$ , there is  $k \in [0, i - 1]$  such that  $M'(k') = M(k)$ .

$M$  and  $M'$  are *bisimilar* if  $M$  is a simulation of  $M'$  and vice versa.

$M'$  is a *contraction* of  $M$  if  $M'$  is of the form  $M(n_0), M(n_1), \dots$ , where  $n_0 < n_1 < \dots$  is an infinite sequence of increasing natural numbers such that there is  $k \geq 0$  so that  $M(n_i) = M(n_{k+1})$  for all  $i \geq k+1$ , and the finite set of positions  $W = \{n_0, \dots, n_k\}$  *minimally* satisfies the following conditions:

- $0, 1 \in W$ ;
- for all  $i \geq 0$ , let  $i_m$  be the smallest position such that  $M(i_m) = M(i)$ . Then,  $i_m \in W$ .

We also say that  $M'$  is a contraction of  $M$  with respect to the sequence  $n_0 < n_1 < \dots$ .

Note that a contraction of a total interpretation over  $P$  is a strongly ultimately periodic total interpretation of size at most  $2 + 2^{|P|}$ . Now, we observe the following.

**Lemma VII.3.** Let  $\varphi \in \text{THT}_1$  and  $M$  and  $M'$  be two interpretations. Then:

- if  $M$  and  $M'$  are bisimilar, then  $M \models \varphi$  iff  $M' \models \varphi$ ;
- if  $M'$  is a contraction of  $M$ , then  $M$  and  $M'$  are bisimilar.

*Proof:* *Property 1:* let  $M$  and  $M'$  be bisimilar. We show that  $M \models \varphi$  iff  $M' \models \varphi$  by induction on the structure of  $\varphi$ . Since  $M(0) = M'(0)$  and  $M(1) = M'(1)$ , the unique non-trivial cases are when  $\varphi$  is either of the form  $\varphi_1 \cup \varphi_2$  or of the form  $\varphi_1 \text{R} \varphi_2$ . For these two cases, we consider the implication  $M \models \varphi \Rightarrow M' \models \varphi$  (the converse implication is symmetric). We crucially use the fact that since  $\varphi \in \text{THT}_1$ , the subformulas  $\varphi_1$  and  $\varphi_2$  have no temporal modalities.

- Case  $\varphi = \varphi_1 \cup \varphi_2$ : let  $M \models \varphi$ . Hence, there exists  $i \geq 0$  such that  $M, i \models \varphi_2$  and  $M, k \models \varphi_1$  for all  $k \in [0, i - 1]$ . Since  $M'$  is a simulation of  $M$ , there exists  $i' \geq 0$  such that  $M'(i') = M(i)$  and for all  $k' \in [0, i' - 1]$ , there is  $k \in [0, i - 1]$  such that  $M'(k') = M(k)$ . Since  $\varphi_1$  and  $\varphi_2$  have no temporal modalities, we obtain that  $M', i' \models \varphi_2$  and  $M', k' \models \varphi_1$  for all  $k' \in [0, i' - 1]$ . Hence,  $M' \models \varphi$ , and the result follows.
- Case  $\varphi = \varphi_1 \text{R} \varphi_2$ : let  $M \models \varphi$ . By the semantics of  $\text{R}$ , there are two cases:
  - $M, i \models \varphi_2$  for all  $i \geq 0$ : since  $M$  is a simulation of  $M'$ , for all  $i' \geq 0$ , there is  $i \geq 0$  such that  $M'(i') = M(i)$ . Thus, since  $\varphi_2$  has no temporal modalities, we obtain that  $M', i \models \varphi_2$  for all  $i \geq 0$ , hence,  $M' \models \varphi$ .
  - There is  $i \geq 0$  such that  $M, i \models \varphi_1 \wedge \varphi_2$  and  $M, k \models \varphi_2$  for all  $k \in [0, i - 1]$ . We proceed as for the case of the until modality.

*Property 2:* let  $M' = M(n_0), M(n_1), \dots$  be a contraction of  $M$  with respect to the sequence  $n_0 < n_1 < \dots$ . We need to show that  $M$  is a simulation of  $M'$  and vice versa. Let  $i \geq 0$ . By construction, there is  $j \geq 0$  such that  $M(n_j) = M(i)$  and  $n_j \leq i$ . Thus, since  $n_0 = 0$ ,  $n_1 = 1$ , and  $M' = M(n_0), M(n_1), \dots$ , we obtain that  $M'$  is a simulation of  $M$ .

Now, we prove that  $M$  is a simulation of  $M'$ . Let  $i' \geq 0$ . We need to show that there is  $i \geq 0$  such that  $M(i) = M'(i')$  and for all  $k \in [0, i - 1]$ , there is  $k' \in [0, i' - 1]$  such that  $M(k) = M'(k')$ . By

construction  $M'(i') = M(n_{i'})$  and one of the following holds:

- for all  $h \geq 0$ , there is  $k' \in [0, i' - 1]$  such that  $M(h) = M'(k')$  (in particular,  $i'$  is a position of the periodic part of  $M'$ ). In this case, by setting  $i = n_{i'}$ , the result follows.
- $n_{i'}$  is the smallest position  $h$  such that  $M(h) = M'(i')$ . We set  $i = n_{i'}$ . Let  $k \in [0, n_{i'} - 1]$  and  $k_m$  be the smallest position such that  $M(k_m) = M(k)$ . Since  $k < n_{i'}$ , by construction,  $k_m = n_h$  for some  $h < i'$  and  $M'(h) = M(n_h)$ . Hence, the result follows. ■

Now, we prove Lemma III.3.

**Lemma III.3.** *Let  $\varphi$  be a  $THT_1$  formula having some equilibrium model. Then, there exists a strongly ultimately periodic equilibrium model of  $\varphi$  of size at most  $2 + 2^{|\varphi|}$ .*

*Proof:* We assume without loss of generality that all the propositions in  $P$  occur in  $\varphi$ . Let  $(T, T)$  be an equilibrium model of  $\varphi$  and  $(T', T')$  be any contraction of  $(T, T)$ . By construction,  $(T', T')$  is a strongly ultimately periodic interpretation of size at most  $2 + 2^{|\varphi|}$ . We show that  $(T', T')$  is an equilibrium model of  $\varphi$ , hence, the result follows. Since  $(T, T)$  is an equilibrium model of  $\varphi$ , by Lemma VII.3,  $(T', T') \models \varphi$ . Now, let  $H' \sqsubset T'$  and  $M' = (H', T')$ . It remains to show that  $M' \models \varphi$ . Let  $n_0 < n_1 < \dots$  be the infinite sequence of increasing natural numbers such that  $(T', T')$  is a contraction of  $(T, T)$  with respect to  $n_0 < n_1 < \dots$ . In particular,  $T' = T(n_0), T(n_1), \dots$ . Let  $M$  be the interpretation defined as follows: for each position  $n_j$  along the sequence  $n_0 < n_1 < \dots$ ,  $M(n_j) = (H'(j), T'(j))$ , and for each position  $i$  which does not occur along the sequence  $n_0 < n_1 < \dots$ ,  $M(i) = (H'(h), T(i))$ , where  $h$  is the smallest position such that  $T(i) = T'(h)$  (since  $(T', T')$  is a contraction of  $(T, T)$  with respect to  $n_0 < n_1 < \dots$  such a  $h$  exists). Evidently,  $M$  is of the form  $(H, T)$  with  $H \sqsubset T$ , and  $M' = M(n_0), M(n_1), \dots$ . Since  $(T', T')$  is a contraction of  $(T, T)$  with respect to  $n_0 < n_1 < \dots$ , one can easily show that  $M$  and  $M'$  are bisimilar. Thus, since  $M \models \varphi$  ( $(T, T)$  is an equilibrium model of  $\varphi$ ), by Lemma VII.3, the result follows. ■

#### F. Proof of Lemma III.4

**Lemma III.4.** *Given  $\varphi \in THT_1$ , the following holds.*

- 1) *Let  $M$  and  $M'$  be two interpretations such that  $M' = M(n_0), M(n_1), \dots$  where  $n_0 < n_1 < \dots$  is an infinite sequence of increasing natural numbers containing all the positions of some witness pattern of  $M$  for  $\varphi$ . Then, for each subformula  $\psi$  of  $\varphi$ ,  $M \models \psi$  iff  $M' \models \psi$ .*
- 2) *Let  $M = (T, T)$  be a total strongly ultimately periodic interpretation satisfying  $\varphi$  of size  $m$ . Then  $M$  is an equilibrium model of  $\varphi$  iff for each  $H \sqsubset T$  such that  $(H, T)$  is a strongly ultimately periodic interpretation of size at most  $m + |\varphi| + 3$ ,  $(H, T) \models \varphi$ .*

*Proof:* *Property 1:* The proof is by induction on the structure of  $\psi$ . The non-trivial cases is when  $\psi$  has an until or release modality as root operator. Hence, either  $\psi = \varphi_1 U \varphi_2$  or  $\psi = \varphi_1 R \varphi_2$  for some formulas  $\varphi_1$  and  $\varphi_2$  which have no temporal modalities. Here, we focus on the case  $\psi = \varphi_1 U \varphi_2$  (the case  $\psi = \varphi_1 R \varphi_2$  being similar). First, assume that  $M \models \varphi_1 U \varphi_2$ . Let  $i$  be the smallest position such that  $M, i \models \varphi_2$ . We have that  $M, k \models \varphi_1$  for all  $k \in [0, i - 1]$ , and by Definition III.3  $i = n_j$  for some  $j \geq 0$ . Thus, since  $M' = M(n_0), M(n_1), \dots$ ,  $n_0 < n_1 < \dots$ , and  $\varphi_1$  and  $\varphi_2$  have no temporal modalities, we obtain that  $M', j \models \varphi_2$  and  $M', h \models \varphi_1$  for all  $h \in [0, j - 1]$ . Hence,  $M' \models \varphi_1 U \varphi_2$ .

Now, assume that  $M \not\models \varphi_1 \mathbf{U} \varphi_2$ . If  $M \not\models F\varphi_2$ , then since  $M' = M(n_0), M(n_1), \dots$  and  $\varphi_2$  has no temporal modalities, we obtain that  $M' \not\models F\varphi_2$ , hence,  $M' \not\models \varphi_1 \mathbf{U} \varphi_2$ . Now, assume that  $M \models F\varphi_2$ . Let  $i$  be the smallest position such that  $M, i \not\models \varphi_1$ . Note that  $M, k \not\models \varphi_2$  for all  $k \in [0, i]$ . By Definition III.3,  $i = n_j$  for some  $j \geq 0$ . Thus, since  $M' = M(n_0), M(n_1), \dots$ ,  $n_0 < n_1 < \dots$ , and  $\varphi_1$  and  $\varphi_2$  have no temporal modalities, we obtain that  $M', j \not\models \varphi_1$  and  $M', h \not\models \varphi_2$  for all  $h \in [0, j]$ . Hence,  $M' \not\models \varphi_1 \mathbf{U} \varphi_2$ , and we are done.

*Property 2:* let  $(T, T)$  be a strongly ultimately periodic interpretation of size  $m$  and  $H \sqsubset T$ . We prove that there is  $H_m \sqsubset T$  such that  $(H_m, T)$  is a strongly ultimately periodic interpretation of size at most  $m + |\varphi| + 3$  and for each subformula  $\psi$  of  $\varphi$ ,  $(H, T) \models \psi$  iff  $(H_m, T) \models \psi$ . Hence, Property 2 follows.

Let  $M = (H, T)$  and  $M_W = (H_W, T_W)$  be a witness extraction of  $M$  for  $\varphi$ . Recall that  $M_W = M(n_0), M(n_1), \dots$ , where  $n_0 < n_1 < \dots$  is a witness pattern of  $M$  for  $\varphi$ . Since  $H \sqsubset T$ , by Definition III.3,  $H_W \sqsubset T_W$ . Let  $j$  be the smallest position such that  $n_j > m$ . Define  $M_m = M(0), \dots, M(m), M(n_j), M(n_{j+1}), \dots$ . Since  $M_W$  is strongly ultimately periodic of size at most  $|\varphi| + 3$  and  $(T, T)$  is strongly ultimately periodic of size  $m$ , it holds that  $M_m$  is a strongly ultimately periodic interpretation of the form  $M_m = (H_m, T)$  having size at most  $m + |\varphi| + 3$  and such that  $H_m \sqsubset T$ . It remains to show that for each subformula  $\psi$  of  $\varphi$ ,  $M \models \psi$  iff  $M_m \models \psi$ . Since  $0 < \dots < m < n_j < n_{j+1}$  contains all the positions of a witness pattern of  $M$  for  $\varphi$ , the result directly follows from Property 1. ■

## VIII. PROOFS FROM SECTION IV

### A. Proof of Theorem IV.1

By the semantics of THT and LTL, the following holds.

**Proposition VIII.1.** *Let  $(H, T)$  be an interpretation and  $\varphi$  be a  $THT^0$  formula. Then,  $(H, T) \models \varphi$  iff  $H \models_{LTL} \varphi$ .*

By Proposition VIII.1, for a  $THT^0$  formula  $\varphi$  and a total interpretation  $(T, T)$ ,  $(T, T)$  is an equilibrium model of  $\varphi$  iff  $T$  is a minimal LTL model of  $\varphi$ . Hence, Theorem IV.1 directly follows from the following result.

**Theorem VIII.1.** *For  $THT^0$  formulas, checking the existence of minimal LTL models is PSPACE-hard.*

Theorem VIII.1 is proved by a polynomial-time reduction from a domino-tiling problem for grids with rows of linear length [22]. An instance  $\mathcal{I} = \langle C, \Delta, n, d_{init}, d_{final} \rangle$  of this problem is as in the proof of Theorem III.1. However, here, a tiling of  $\mathcal{I}$  is defined as a mapping  $f : [0, k] \times [0, n-1] \rightarrow \Delta$ , i.e., the number of columns is  $n$ . It is well-known that checking the existence of a tiling for  $\mathcal{I}$  is PSPACE-complete [22]. We construct in polynomial time a  $THT^0(X, F, G)$  formula  $\varphi_{\mathcal{I}}$  which admits a minimal LTL model iff there exists a tiling of  $\mathcal{I}$ . Hence, Theorem VIII.1 follows.

*Encoding of tilings:* We use the set  $P$  of atomic propositions given by  $P = \{1, \dots, n\} \times \Delta$ . Rows of tilings are encoded by finite words of the form  $\{(1, d_1)\} \dots \{(n, d_n)\}$ , and a tiling  $f$  is encoded by the finite word  $w$  over  $2^P$  corresponding to the sequence of row encodings of  $f$ , starting from the first row of  $f$ .

*Construction of  $\varphi_{\mathcal{I}}$ :* fix an LTL interpretation  $T$  over  $P$ . The LTL interpretation  $T$  is *well-formed* if for every position  $i \geq 0$ ,  $T(i)$  is a singleton.  $T$  is *almost well-formed* if there exists a suffix of  $T$  which is well-formed. First, we observe the following.

**Lemma VIII.1.** *One can construct in polynomial time a  $THT^0(X, F, G)$  formula  $\psi_{\mathcal{I}}$  such that for all LTL interpretations  $T$  which are almost well-formed, the following holds:*

- $T \models \psi_{\mathcal{I}}$  iff some suffix of  $T$  is of the form  $w_0 \cdot w_1 \cdot \dots$  such that  $w_i$  encodes a tiling for all  $i \geq 0$  (i.e., some suffix of  $T$  is the  $\omega$ -concatenation of tiling encodings).

*Proof:* The  $\text{THT}^0(\mathbf{X}, \mathbf{F}, \mathbf{G})$  formula  $\psi_{\mathcal{I}}$  is defined as follows.

$$\begin{aligned} \psi_{\mathcal{I}} = & \underbrace{\mathbf{GF}((n, d_{\text{final}}) \wedge \mathbf{X}(1, d_{\text{init}}))}_{\text{initialization and acceptance}} \wedge \mathbf{FG} \left\{ \right. \\ & \underbrace{\left( \bigvee_{d, d' \in \Delta} \left[ ((n, d) \wedge \mathbf{X}(1, d')) \vee \bigvee_{i=1}^{i=n-1} ((i, d) \wedge \mathbf{X}(i+1, d')) \right] \right)}_{\text{there is a suffix which is a sequence of row encodings}} \wedge \\ & \underbrace{\left( \bigvee_{d \in \Delta} \left[ (n, d) \vee \bigvee_{i=1}^{i=n-1} \bigvee_{d' \in \Delta: (d')_{\text{left}} = d_{\text{right}}} ((i, d) \wedge \mathbf{X}(i+1, d')) \right] \right)}_{\text{adjacent-row requirement}} \wedge \\ & \underbrace{\left( \bigvee_{j=0}^{n-1} \mathbf{X}^j [(n, d_{\text{final}}) \wedge \mathbf{X}(1, d_{\text{init}})] \vee \bigvee_{d, d' \in \Delta: (d')_{\text{down}} = d_{\text{up}}} \bigvee_{i=1}^{i=n} [(i, d) \wedge \mathbf{X}^n(i, d')] \right)}_{\text{adjacent-column requirement}} \left. \right\} \end{aligned}$$

Let  $\psi_{\mathcal{I}}$  be the  $\text{THT}^0(\mathbf{X}, \mathbf{F}, \mathbf{G})$  formula of Lemma VIII.1 and  $\psi_{\text{no\_cell}}$  be the propositional  $\text{THT}^0$  formula given by  $\bigvee_{p, p' \in P: p \neq p'} (p \wedge p')$ . Then, the  $\text{THT}^0(\mathbf{X}, \mathbf{F}, \mathbf{G})$  formula  $\varphi_{\mathcal{I}}$  is defined as follows: ■

$$\varphi_{\mathcal{I}} = \mathbf{G} \left( \bigvee_{p \in P} p \right) \wedge \left( \mathbf{GF}(\psi_{\text{no\_cell}}) \vee \psi_{\mathcal{I}} \right)$$

Correctness of the construction directly follows from the following lemma, which concludes the proof of Theorem VIII.1.

**Lemma VIII.2.** *There is a tiling of  $\mathcal{I}$  iff there is a minimal LTL model of  $\varphi_{\mathcal{I}}$ .*

*Proof:* First, assume that there exists a minimal LTL model  $\mathbf{T}$  of  $\varphi_{\mathcal{I}}$ . By construction of  $\varphi_{\mathcal{I}}$ , for all positions  $i \geq 0$ ,  $\mathbf{T}(i) \neq \emptyset$ . Hence,  $\mathbf{T}$  is almost well-formed iff  $\mathbf{T} \not\models_{\text{LTL}} \mathbf{GF}(\psi_{\text{no\_cell}})$ . We claim that  $\mathbf{T}$  is almost well-formed. We assume the contrary and derive a contradiction. Hence,  $\mathbf{T} \models_{\text{LTL}} \mathbf{GF}(\psi_{\text{no\_cell}})$ . This implies that there exists  $\mathbf{H} \sqsubset \mathbf{T}$  such that  $\mathbf{H} \models \mathbf{GF}(\psi_{\text{no\_cell}})$  and for all positions  $i \geq 0$ ,  $\mathbf{H}(i) \neq \emptyset$ . By construction of  $\varphi_{\mathcal{I}}$ , we obtain that  $\mathbf{H} \models_{\text{LTL}} \varphi_{\mathcal{I}}$  which contradicts the minimality of  $\mathbf{T}$ . Thus, the claim holds, and by construction of  $\varphi_{\mathcal{I}}$ ,  $\mathbf{T}$  is almost well-formed and  $\mathbf{T} \models \psi_{\mathcal{I}}$ . By Lemma VIII.1, some suffix of  $\mathbf{T}$  is the  $\omega$ -concatenation of tiling encodings. Hence, there exists a tiling of  $\mathcal{I}$ .

For the converse implication, assume that there exists a tiling  $f$  of  $\mathcal{I}$ . Let  $\mathbf{T}$  be the LTL interpretation given by  $(w_f)^\omega$  where  $w_f$  is the encoding of  $f$ . Note that  $\mathbf{T}$  is well-formed and by Lemma VIII.1,  $\mathbf{T}$  is an LTL model of  $\varphi_{\mathcal{I}}$ . Moreover, since  $\mathbf{T}$  is well-formed, for all  $\mathbf{H} \sqsubset \mathbf{T}$ , there exists a position  $i$  such that  $\mathbf{H}(i) = \emptyset$ . Hence, by construction of  $\varphi_{\mathcal{I}}$ ,  $\mathbf{H} \not\models_{\text{LTL}} \varphi_{\mathcal{I}}$ . Thus,  $\mathbf{T}$  is a minimal LTL model of  $\varphi_{\mathcal{I}}$  and we are done. ■

### B. Proof of Theorem IV.3 for the fragment $\text{THT}_1$

**Theorem VIII.2.** *Let  $\varphi$  be a  $\text{THT}_1$  formula which is LTL satisfiable. Then, there exists a minimal LTL model of  $\varphi$ .*

*Proof:* First, we need additional definitions. A  $\text{THT}_1$  formula  $\varphi$  is in disjunctive normal form if  $\varphi$  is of the form  $D_1 \vee \dots \vee D_k$ , where for all  $i \in [1, k]$ ,  $D_i$ , called *main disjunct* of  $\varphi$ , is of the form

$$\eta \wedge (\text{X}\chi) \wedge (\text{G}\psi) \wedge (\xi_1 \text{U} \phi_1) \wedge \dots \wedge (\xi_m \text{U} \phi_m)$$

where  $\eta$  has no temporal modalities. Since  $\varphi_1 \text{R} \varphi_2$  can be seen as a shorthand for  $(\varphi_2 \text{U} (\varphi_1 \wedge \varphi_2)) \vee \text{G}\varphi_2$ , given a  $\text{THT}_1$  formula  $\varphi$ , one can construct a  $\text{THT}_1$  formula  $\psi$  in disjunctive normal form such that for all LTL interpretations  $\text{T}$  and positions  $i \geq 0$ ,  $\text{T}, i \models_{\text{LTL}} \varphi$  iff  $\text{T}, i \models_{\text{LTL}} \psi$ . Thus, without loss of generality, we can assume that the given LTL satisfiable formula  $\varphi$  in  $\text{THT}_1$  is in disjunctive normal form.

Let  $D_1, \dots, D_k$  be the main disjuncts of  $\varphi$  and  $\text{T}$  be an LTL model of  $\varphi$ . Hence, there exists  $i \in [1, k]$  such that  $\text{T} \models_{\text{LTL}} D_i$ . We claim that there exists  $\text{T}_i \sqsubseteq \text{T}$  such that  $\text{T}_i$  is a minimal LTL model of  $D_i$ . Before proving this, we first observe that the claim implies the existence of a minimal LTL model of  $\varphi$ . Indeed, if there exists  $j \neq i$  and  $\text{T}_j \sqsubset \text{T}_i$  such that  $\text{T}_j \models_{\text{LTL}} D_j$ , by applying the claim, there must exist  $\text{T}'_j \sqsubseteq \text{T}_j$  such that  $\text{T}'_j$  is a minimal LTL model of  $D_j$  and for all  $\text{T}'' \sqsubseteq \text{T}'_j$ ,  $\text{T}''$  is not an LTL model of  $D_i$ . Thus, by iterating the reasoning to the remaining set  $\{D_1, \dots, D_k\} \setminus \{D_i, D_j\}$  of main disjuncts, the existence of a minimal LTL model of  $\varphi$  follows. Now, we prove the claim. The main disjunct  $D_i$  is of the form

$$\eta \wedge (\text{X}\chi) \wedge (\text{G}\psi) \wedge (\xi_1 \text{U} \phi_1) \wedge \dots \wedge (\xi_m \text{U} \phi_m)$$

where  $\eta$  has no temporal modalities. Moreover, since  $D_i \in \text{THT}_1$ , the subformulas  $\chi, \psi, \xi_1, \phi_1, \dots, \xi_m, \phi_m$  have no temporal modalities. Since  $\text{T} \models_{\text{LTL}} D_i$ , for all  $j \in [1, m]$ , there exists the smallest position  $\ell_j$  such that  $\text{T}, \ell_j \models_{\text{LTL}} \phi_j$  (note that  $\text{T}, h \models_{\text{LTL}} \xi_j$  for all  $h \in [0, \ell_j - 1]$ ). Let  $\ell = \max(\{\ell_1, \dots, \ell_m, 1\})$  and  $\text{T}'$  be the LTL interpretation defined as follows:

- for all  $n \geq 0$ ,  $\text{T}'(n) = \text{T}(n)$  if  $n \leq \ell$ ; otherwise,  $\text{T}'(n)$  is a minimal subset of  $\text{T}(n)$  such that  $\text{T}'(n)$  satisfies the propositional formula  $\psi$ .

By construction  $\text{T}' \sqsubseteq \text{T}$  and since  $\eta, \chi, \psi, \xi_1, \phi_1, \dots, \xi_m, \phi_m$  are propositional formulas,  $\text{T}'$  is an LTL model of  $D_i$ . Moreover, for all LTL interpretations  $\text{T}''$  such that  $\text{T}''(n) \subset \text{T}'(n)$  for some  $n > \ell$ ,  $\text{T}'' \not\models_{\text{LTL}} D_i$ . Hence, the set of LTL interpretations  $\text{T}''$  such that  $\text{T}'' \sqsubset \text{T}'$  and  $\text{T}''$  is an LTL model of  $D_i$  is finite. Thus, since  $\text{T}' \sqsubseteq \text{T}$ , there exists a minimal LTL model  $\text{T}_i$  of  $D_i$  such that  $\text{T}_i \sqsubseteq \text{T}$ , and we are done. ■

### C. Full proof of Lemma IV.3

**Lemma IV.3.** *Let  $\varphi$  be a  $\text{THT}(X, U)$  formula and  $M = (\text{T}, \text{T})$  be an equilibrium model of  $\varphi$ . Then,  $M$  is almost-empty.*

*Proof:* Let  $\varphi$  and  $M = (\text{T}, \text{T})$  be as in the statement of the lemma. We assume without loss of generality that  $\varphi$  is *not* of the form  $\psi_1 \text{U} \psi_2$  (otherwise, we consider the formula  $(\psi_1 \text{U} \psi_2) \wedge \top$ ).



Fix a set of witnesses  $W$  of  $M$  for  $\varphi$ . Let  $\ell$  be the greatest position occurring in  $W$ . We define an LTL interpretation  $H_W \sqsubseteq T$  as follows:

- for all  $i \geq 0$ ,  $H_W(i) = T(i)$  if  $i \leq \ell + d_X(\varphi)$ , and  $H_W(i) = \emptyset$  otherwise.

We show that  $H_W = T$ , hence,  $M = (T, T)$  is almost empty, and the result follows. For this, since  $M = (T, T)$  is an equilibrium model of  $\varphi$ , it suffices to prove that  $(H_W, T), 0 \models \varphi$ . Since  $(0, \varphi) \in W$ , the result directly follows from the following claim:

*Claim:* for all  $(j, \psi) \in W$  and subformulas  $\xi$  of  $\psi$ , the following holds:

- 1) for all  $k \in [0, d_X(\psi)]$  such that  $d_X(\xi) \leq d_X(\psi) - k$ ,  $(T, T), j + k \models \xi$  iff  $(H_W, T), j + k \models \xi$ .
- 2) for all  $k \in [0, j]$ ,  $(T, T), k \models \xi$  iff  $(H_W, T), k \models \xi$ ;

*Proof of the claim:* Let  $(j, \psi) \in W$  and  $\xi$  be a subformula of  $\psi$ . We prove Properties 1 and 2 by induction on the structure of  $\xi$ . We only consider the cases where  $\xi$  has a temporal modality as root operator (the other cases easily follow from the construction and induction hypothesis). Thus, since  $\xi$  is a  $\text{THT}(X, U)$  formula, either  $\xi = X\xi_1$  or  $\xi = \xi_1 \cup \xi_2$  for some  $\text{THT}(X, U)$  formulas  $\xi_1$  and  $\xi_2$ . We prove the implication  $(T, T), j + k \models \xi \Rightarrow (H_W, T), j + k \models \xi$  of Property 1, and the implication  $(T, T), k \models \xi \Rightarrow (H_W, T), k \models \xi$  of Property 2 (since the converse implications directly follow from Proposition II.1(1)).

*Property 1:* let  $(T, T), j + k \models \xi$ , where  $k \in [0, d_X(\psi)]$  and  $d_X(\xi) \leq d_X(\psi) - k$ . If  $\xi = X\xi_1$ , then  $(T, T), j + (k + 1) \models \xi_1$ ,  $k < d_X(\psi)$ , and  $d_X(\xi_1) \leq d_X(\psi) - (k + 1)$ . Hence, by applying the induction hypothesis, Property 1 follows.

Now, assume that  $\xi = \xi_1 \cup \xi_2$ . First, we consider the case when  $\xi = \psi$ . Since  $d_X(\xi) \leq d_X(\psi) - k$ , it follows that  $k = 0$ . Since  $(j, \xi_1 \cup \xi_2) \in W$  and  $\varphi \neq \xi_1 \cup \xi_2$ , by Definition IV.1, we have that  $(T, T), j \models \xi_2$ . Hence, by applying the induction hypothesis for Property 1, the result follows.

Now, assume that  $\xi_1 \cup \xi_2$  is a strict subformula of  $\psi$ . Since  $(T, T), j + k \models \xi_1 \cup \xi_2$  and  $(j, \psi) \in W$ , by Definition IV.1, for some position  $j'$ ,  $(j', \xi_1 \cup \xi_2) \in W$  and  $(T, T), j' \models \xi_2$ . Moreover, either  $\xi_1 \cup \xi_2 \in \text{Fin}(\varphi, M)$  and  $j'$  is the greatest position such that  $(T, T), j' \models \xi_2$ , or  $\xi_1 \cup \xi_2 \in \text{Inf}(\varphi, M)$  and  $j' > j + d_X(\varphi) \geq j + k$ . Hence,  $j' \geq j + k$ . Thus, since  $(T, T), j + k \models \xi_1 \cup \xi_2$  and  $(T, T), j' \models \xi_2$ , there must be  $\ell \in [j + k, j']$  such that  $(T, T), \ell \models \xi_2$  and  $(T, T), m \models \xi_1$  for all  $m \in [j + k, \ell - 1]$ . Since  $(j', \xi_1 \cup \xi_2) \in W$ , by applying the induction hypothesis on Property 2 for the subformulas  $\xi_1$  and  $\xi_2$  of  $\xi_1 \cup \xi_2$ , the result follows.

*Property 2:* for the case  $\xi = X\xi_1$ , Property 2 directly follows from Property 1 and the induction hypothesis. Now, let us consider the case  $\xi = \xi_1 \cup \xi_2$ . Let  $(T, T), k \models \xi$  with  $k \in [0, j]$ . First, assume that  $\xi_1 \cup \xi_2 = \psi$ . Since  $(j, \xi_1 \cup \xi_2) \in W$  and  $\varphi \neq \xi_1 \cup \xi_2$ , by Definition IV.1, we have that  $(T, T), j \models \xi_2$ . Thus, since  $(T, T), k \models \xi$  and  $k \in [0, j]$ , there must be  $\ell \in [k, j]$  such that  $(T, T), \ell \models \xi_2$  and  $(T, T), m \models \xi_1$  for all  $m \in [k, \ell - 1]$ . Since  $(j, \xi_1 \cup \xi_2) \in W$ , by applying the induction hypothesis on Property 2 for the subformulas  $\xi_1$  and  $\xi_2$  of  $\xi_1 \cup \xi_2$ , the result follows. Now, assume that  $\xi_1 \cup \xi_2$  is a strict subformula of  $\psi$ . Since  $(T, T), k \models \xi_1 \cup \xi_2$  and  $(j, \psi) \in W$ , by Definition IV.1, for some position  $j'$ ,  $(j', \xi_1 \cup \xi_2) \in W$  and  $(T, T), j' \models \xi_2$ . Moreover, either  $\xi_1 \cup \xi_2 \in \text{Fin}(\varphi, M)$  and  $j'$  is the greatest position such that  $(T, T), j' \models \xi_2$ , or  $\xi_1 \cup \xi_2 \in \text{Inf}(\varphi, M)$  and  $j' > j + d_X(\varphi) \geq k$ . Hence,  $j' \geq k$ . Thus, since  $(T, T), k \models \xi_1 \cup \xi_2$  and  $(T, T), j' \models \xi_2$ , there must be  $\ell \in [k, j']$  such that  $(T, T), \ell \models \xi_2$  and  $(T, T), m \models \xi_1$  for all  $m \in [k, \ell - 1]$ . Since  $(j', \xi_1 \cup \xi_2) \in W$ , by applying the induction hypothesis on Property 2 for the subformulas  $\xi_1$  and  $\xi_2$  of  $\xi_1 \cup \xi_2$ , the result follows. ■

#### D. Full proof of Lemma IV.4

**Lemma IV.4.** *Let  $\varphi$  be a  $\text{THT}(X, F)$  formula and  $M = (T, T)$  be an equilibrium model of  $\varphi$ . Then,  $M$  has at most  $d_X(\varphi) \cdot (|\varphi| + 1)$  non-empty positions.*

*Proof:* We assume without loss of generality that  $\varphi$  is *not* of the form  $F\psi$  (otherwise, we consider the formula  $(F\psi) \wedge \top$ ). Let  $W$  be a set of witnesses of  $M$  for  $\varphi$  according to Definition IV.1. By Definition IV.1,  $W$  has cardinality at most  $|\varphi| + 1$ . Now, we define an LTL interpretation  $H_W \sqsubseteq T$  as follows:

- for all  $i \geq 0$ , if there is  $(j, \psi) \in W$  such that  $j \leq i$  and  $i - j \leq d_X(\varphi)$ , then  $H_W(i) = T(i)$ ; otherwise,  $H_W(i) = \emptyset$ .

By construction, the set of non-empty positions of the interpretation  $(H_W, T)$  has cardinality at most  $d_X(\varphi) \cdot (|\varphi| + 1)$ . We show that  $H_W = T$ , hence, the result follows. For this, since  $M = (T, T)$  is an equilibrium model of  $\varphi$ , it suffices to prove that  $(H_W, T), 0 \models \varphi$ . Since  $(0, \varphi) \in W$ , the result directly follows from the following claim:

*Claim:* for all  $(i, \psi) \in W$ ,  $k \in [0, d_X(\psi)]$ , and subformulas  $\xi$  of  $\psi$  such that  $d_X(\xi) \leq d_X(\psi) - k$ ,  $(T, T), i + k \models \xi$  iff  $(H_W, T), i + k \models \xi$ .

*Proof of the claim:* Let  $(i, \psi) \in W$ ,  $k \in [0, d_X(\psi)]$ , and  $\xi$  be a subformula of  $\psi$  such that  $d_X(\xi) \leq d_X(\psi) - k$ . The implication  $(H_W, T), i + k \models \xi \Rightarrow (T, T), i + k \models \xi$  directly follows from Proposition II.1(1). For the converse implication, assume that  $(T, T), i + k \models \xi$ . We show that  $(H_W, T), i + k \models \xi$  by induction on the structure of  $\xi$ . We only consider the cases where  $\xi$  has a temporal modality as root operator (the other cases easily follow from the induction hypothesis and the fact that by construction  $H_W(i + k) = T(i + k)$ ). Thus, since  $\xi$  is a  $\text{THT}(X, F)$  formula, either  $\xi = X\xi'$  or  $\xi = F\xi'$ . First, assume that  $\xi = X\xi'$ . Hence,  $(T, T), i + (k + 1) \models \xi'$ . Since  $\xi$  is a subformula of  $\psi$  such that  $d_X(\xi) \leq d_X(\psi) - k$  and  $k \in [0, d_X(\psi)]$ , we have that  $k < d_X(\psi)$  and  $d_X(\xi') \leq d_X(\psi) - (k + 1)$ . By the induction hypothesis,  $(H_W, T), i + (k + 1) \models \xi'$ , hence,  $(H_W, T), i + k \models \xi$ , and the result follows.

Now, assume that  $\xi = F\xi'$ . First, assume that  $\xi = \psi$ . Since  $d_X(\xi) \leq d_X(\psi) - k$ , it follows that  $k = 0$ . Since  $(i, F\xi') \in W$  and  $\varphi \neq F\xi'$ , by Definition IV.1, we have that  $(T, T), i \models \xi'$ . Hence, by applying the induction hypothesis, the result follows. Now, assume that  $F\xi'$  is a strict subformula of  $\psi$ . Since  $(T, T), i + k \models F\xi'$ , there exists  $j \geq i + k$  such that  $(T, T), j \models \xi'$ . We need to show that  $(H_W, T), i + k \models F\xi'$ . By construction,  $F\xi' \in \text{Fin}(\varphi, M) \cup \text{Inf}(\varphi, M)$ . We distinguish two cases:

- $F\xi' \in \text{Fin}(\varphi, M)$ : since  $(T, T), j \models \xi'$ , by Definition IV.1, there exists the greatest position  $j'$  such that  $(T, T), j' \models \xi'$  and  $(j', F\xi') \in W$ . Hence,  $j' \geq j$ . Moreover, by applying the induction hypothesis, we have that  $(H_W, T), j' \models \xi'$ . Thus, since  $j \geq i + k$  and  $j' \geq j$ , we obtain that  $(H_W, T), i + k \models F\xi'$ , and the result holds.
- $F\xi' \in \text{Inf}(\varphi, M)$ : by Definition IV.1, there exists a position  $m$  such that  $(m, F\xi') \in W$  and  $(T, T), m \models \xi'$ . Hence, by applying the induction hypothesis,  $(H_W, T), m \models \xi'$ . Moreover, since  $(i, \psi) \in W$ ,  $F\xi'$  is a strict subformula of  $\psi$ , and  $k \leq d_X(\varphi)$ , by Definition IV.1, it follows that  $m > i + k$ . Hence,  $(H_W, T), i + k \models F\xi'$ , and the result follows, which concludes. ■

### E. Proof of Lemma IV.5

**Lemma IV.5.** *Let  $\varphi$  be a  $\text{THT}(X, F)$  formula,  $n \geq 1$ , and  $M = (T, T)$  be an equilibrium model of  $\varphi$  having  $n$  non-empty positions. Then, there exists an almost-empty equilibrium model of  $\varphi$  of size at most  $n \cdot (d_X(\varphi) + 1)$ .*

*Proof:* By hypothesis  $M$  is an almost-empty equilibrium model of  $\varphi$  having  $n$  non-empty positions. Let  $\ell$  be the size of  $M$ . If  $\ell \leq n \cdot (d_X(\varphi) + 1)$ , we are done. Otherwise, we show that there exists an almost-empty equilibrium model of  $\varphi$  of size  $\ell - 1$  and having  $n$  non-empty positions. Hence, by iterating the reasoning, the result follows. Since  $\ell > n \cdot (d_X(\varphi) + 1)$ , there must be a set of empty positions of  $M$  of the form  $[h, k]$  such that  $k \leq \ell$  and  $k - h > d_X(\varphi) + 1$ . Let  $M'$  be the total interpretation defined as follows: for all  $i \geq 0$ ,  $M'(i) = M(i)$  if  $i < k$ , and  $M'(i) = M(i + 1)$  otherwise. Intuitively,  $M'$  is obtained from  $M$  by contracting the interval  $[h, k]$  of one position. Note that  $M'$  is an almost-empty total interpretation of size  $\ell - 1$  and having  $n$  non-empty positions. One can easily show that  $M'$  is still an equilibrium model of  $\varphi$ , which concludes. ■